J. London Math. Soc. (2) 103 (2021) 558–586 doi:10.1112/jlms.12384

A semicircle law and decorrelation phenomena for iterated Kolmogorov loops

Karen Habermann

Abstract

We consider a standard one-dimensional Brownian motion on the time interval [0,1] conditioned to have vanishing iterated time integrals up to order N. We show that the resulting processes can be expressed explicitly in terms of shifted Legendre polynomials and the original Brownian motion, and we use these representations to prove that the processes converge weakly as $N \to \infty$ to the zero process. This gives rise to a polynomial decomposition for Brownian motion. We further study the fluctuation processes obtained through scaling by $\sqrt{N}$ and show that they converge in finite-dimensional distributions as $N \to \infty$ to a collection of independent zero-mean Gaussian random variables whose variances follow a scaled semicircle. The fluctuation result is a consequence of a limit theorem for Legendre polynomials which quantifies their completeness and orthogonality property. In the proof of the latter, we encounter a Catalan triangle.

1. *Introduction*

Let $(B_t)_{t\in[0,1]}$ be a Brownian motion in $\mathbb{R}$, which we assume is realised as the coordinate process on the path space $\{w \in C([0,1],\mathbb{R})\colon w_0 = 0\}$ under Wiener measure $\mathbb{P}$. The stochastic process in $\mathbb{R}^2$ which pairs the standard one-dimensional Brownian motion $(B_t)_{t\in[0,1]}$ with its time integral is the Kolmogorov diffusion, named after Kolmogorov [14]. Similarly, pairing Brownian motion with its iterated time integrals up to some order gives rise to the iterated Kolmogorov diffusion.

Definition 1.1. Let $N \in \mathbb{N}$. The stochastic process $(\boldsymbol{B}_t^N)_{t\in[0,1]}$ in $\mathbb{R}^N$ defined by

$$\boldsymbol{B}_t^N = \left(B_t, \int_0^t B_{s_1}\,\mathrm{d}s_1, \int_0^t\int_0^{s_2} B_{s_1}\,\mathrm{d}s_1\,\mathrm{d}s_2, \dots, \int_0^t\int_0^{s_{N-1}}\cdots\int_0^{s_2} B_{s_1}\,\mathrm{d}s_1\dots\mathrm{d}s_{N-1}\right)$$

is the iterated Kolmogorov diffusion of step N.

In particular, $(\boldsymbol{B}_t^1)_{t\in[0,1]}$ is simply the Brownian motion $(B_t)_{t\in[0,1]}$ and $(\boldsymbol{B}_t^2)_{t\in[0,1]}$ is the associated Kolmogorov diffusion. Since $(\boldsymbol{B}_t^N)_{t\in[0,1]}$ is a Gaussian process for all $N \in \mathbb{N}$, we can make sense of conditioning the process $(\boldsymbol{B}_t^N)_{t\in[0,1]}$ on $\boldsymbol{B}_1^N = 0$. Considering the first component of the resulting process shows the existence of the iterated Kolmogorov loop of step N.

Definition 1.2. The iterated Kolmogorov loop of step $N \in \mathbb{N}$ is the stochastic process in $\mathbb{R}$ obtained by conditioning $(B_t)_{t\in[0,1]}$ on $\boldsymbol{B}_1^N = 0$.

Received 16 September 2019; published online 16 September 2020.

2010 *Mathematics Subject Classification* 60F05 (primary), 33C45 (secondary).

This research was supported by the German Research Foundation DFG through the Hausdorff Center for Mathematics under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813.

The terminology is motivated by Baudoin [4, Section 3.6] where Brownian motion in $\mathbb{R}^d$ conditioned to have trivial truncated signature of order N is called the Brownian loop of step N. Note that this is not to be confused with Brownian loops appearing in the context of Schramm–Loewner evolutions, cf. Lawler and Werner [16].

We study the iterated Kolmogorov loops of step N in the limit $N \to \infty$. Our analysis exploits the explicit expression below for iterated Kolmogorov loops in terms of shifted Legendre polynomials. For the proof and further discussions, see Section 3.

PROPOSITION 1.3. *Let Q_n be the shifted Legendre polynomial of degree n on* $[0,1]$. *For $N \in \mathbb{N}$, the stochastic process $(L_t^N)_{t\in[0,1]}$ in $\mathbb{R}$ defined by*

$$L_t^N = B_t - \sum_{n=0}^{N-1}(2n+1)\int_0^t Q_n(r)\,\mathrm{d}r\int_0^1 Q_n(r)\,\mathrm{d}B_r \tag{1.1}$$

has the same law as the iterated Kolmogorov loop of step N.

As a consequence of the completeness and orthogonality of the shifted Legendre polynomials, we obtain a law of large numbers type theorem for the iterated Kolmogorov loops. This result can be rephrased to give a polynomial decomposition of Brownian motion, cf. Section 3.

THEOREM 1.4. *Let $\Omega^{0,0} = \{w \in C([0,1],\mathbb{R})\colon w_0 = w_1 = 0\}$ be the set of continuous loops in $\mathbb{R}$ at zero. The laws of the iterated Kolmogorov loops of step N converge weakly on $\Omega^{0,0}$ as $N \to \infty$ to the unit mass δ_0 at the zero path.*

A similar question can be posed for Brownian loops. We conjecture that, for $d \geqslant 2$, the Brownian loops of step N converge weakly to the zero process in $\mathbb{R}^d$ as $N \to \infty$, see [11, Conjecture 4.1.3].

Going beyond the law of large numbers, we further study the fluctuation processes of the iterated Kolmogorov loops of step N obtained through scaling by $\sqrt{N}$ in the limit $N \to \infty$. As seen in Section 3, cf. Lemma 3.2, the covariance function C_N of the iterated Kolmogorov loop of step N is given, for $s, t \in [0,1]$, by

$$C_N(s,t) = \min(s,t) - \sum_{n=0}^{N-1}(2n+1)\int_0^s Q_n(r)\,\mathrm{d}r\int_0^t Q_n(r)\,\mathrm{d}r\,.$$

Our central limit type theorem for the iterated Kolmogorov loops then relies on the following limit theorem involving Legendre polynomials. For convenience, it is expressed in terms of the Legendre polynomials on $[-1,1]$.

THEOREM 1.5. *Let P_n be the Legendre polynomial of degree n on $[-1,1]$. Fix $x, y \in [-1,1]$ and, for $N \in \mathbb{N}$, set*

$$R_N(x,y) = N\left(\min(1+x,1+y) - \sum_{n=0}^{N-1}\frac{2n+1}{2}\int_{-1}^x P_n(z)\,\mathrm{d}z\int_{-1}^y P_n(z)\,\mathrm{d}z\right). \tag{1.2}$$

Then, we have

$$\lim_{N\to\infty} R_N(x,y) = \begin{cases} \dfrac{1}{\pi}\sqrt{1-x^2} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases},$$

that is, $R_N\colon [-1,1]\times[-1,1] \to \mathbb{R}$ converges pointwise as $N \to \infty$ to the specified limit function.

This result quantifies an integrated version of the completeness and orthogonality property for the Legendre polynomials, which in terms of the Dirac delta function is stated as, for $x, y \in [-1, 1]$,

$$\sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) P_n(y) = \delta(x-y) .$$

The proof of Theorem 1.5 is split into an on-diagonal and an off-diagonal analysis. The pointwise convergence on the diagonal follows from a convergence of moments, cf. Proposition 4.9, and a locally uniform convergence implied by Lemma 5.3 and the Arzelà–Ascoli theorem, whereas the pointwise convergence away from the diagonal relies on a Christoffel–Darboux type formula for the integrals of the Legendre polynomials, cf. Proposition 5.1. In both parts, we use asymptotic estimates for Legendre polynomials and their integrals which are implied by the Darboux formula for Jacobi polynomials. For convenience, we include the Darboux formula as Theorem 2.4.

Equipped with Theorem 1.5, we deduce a central limit type theorem for the iterated Kolmogorov loops. With Proposition 1.3 in mind, we consider the processes $(F_t^N)_{t\in[0,1]}$ defined by $F_t^N = \sqrt{N} L_t^N$.

THEOREM 1.6. *The fluctuation processes $(F_t^N)_{t\in[0,1]}$ converge in finite-dimensional distributions as $N \to \infty$ to the collection $(F_t)_{t\in[0,1]}$ of independent zero-mean Gaussian random variables whose variances are given, for $t \in [0,1]$, by*

$$\mathbb{E}\left[F_t^2\right] = \frac{1}{\pi}\sqrt{t(1-t)} .$$

It is certainly interesting that the variances of the limit fluctuations follow a scaled semicircle, and we remark that semicircles naturally appear in other limit theorems such as the Wigner semicircle law in random matrix theory, cf. [**1**, Theorem 2.1.1], or the central limit theorem in free probability, see [**22**, Theorem 8.10]. Moreover, as pointed out in Remark 5.4, we can obtain a non-trivial bound on the scale of the decorrelation observed for the rescaled iterated Kolmogorov loops.

The reason for considering convergence in finite-dimensional distributions in Theorem 1.6 is that while the collection $(F_t)_{t\in[0,1]}$ of independent zero-mean Gaussian random variables is well defined, see [**6**, Section 2.3], it neither has a realisation as a process in $C([0,1],\mathbb{R})$, cf. [**13**, Example 1.2.4], nor is it equivalent to a measurable process, cf. [**13**, Example 1.2.5]. This is also why $(F_t)_{t\in[0,1]}$, which could be thought of as an inhomogeneous white noise process with vanishing power spectral density, is not treated as a useful mathematical model for white noise.

The paper is organised as follows. In Section 2, we recall properties of Legendre polynomials and their integrals, and we introduce complex-valued polynomials which simplify the presentation and some of the arguments given in Section 4. That section is concerned with studying the moments of R_N on the diagonal in the limit $N \to \infty$. As part of the analysis, which uses partial fraction decompositions, we encounter a Catalan triangle, see Remark 4.5. In Section 3, we determine an expression for the iterated Kolmogorov loop of step N in terms of the inverse of an $N \times N$ factorial Hankel matrix, and we prove Proposition 1.3 as well as Theorem 1.4. In Section 5, we give the proof of Theorem 1.5 which makes use of the Christoffel–Darboux type formula for the integrals of the Legendre polynomials stated in Proposition 5.1, and we conclude with the proof of Theorem 1.6. Throughout, we use the convention that $\mathbb{N}$ denotes the positive integers, whereas $\mathbb{N}_0$ refers to the non-negative integers.

2. *Legendre polynomials and their integrals*

We discuss properties of Legendre polynomials that are needed for our subsequent analysis and we extend the Legendre polynomials to a family of complex-valued polynomials on $[-1, 1]$.

Using this extension, we introduce a second family of complex-valued polynomials, which is linked to the integrals of Legendre polynomials.

Let $\{P_n : n \in \mathbb{N}_0\}$ be the family of the Legendre polynomials on the interval $[-1,1]$. Following the physical motivation presented in Arfken and Weber [**3**, Section 12.1] of considering the electrostatic potential of a point charge, the Legendre polynomials can be defined by means of a generating function through

$$\sum_{n=0}^{\infty} P_n(x)z^n = \frac{1}{\sqrt{1-2xz+z^2}} \quad \text{for } z \in (-1,1)\ .$$

As derived in [**3**, Section 12.2], the generating function can be used to establish the Bonnet recursion formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad \text{for } n \in \mathbb{N} \text{ and } x \in [-1,1] \tag{2.1}$$

as well as the relation

$$(2n+1)\int_{-1}^{x} P_n(z)\,\mathrm{d}z = P_{n+1}(x) - P_{n-1}(x) \quad \text{for } n \in \mathbb{N} \text{ and } x \in [-1,1]\ . \tag{2.2}$$

It is further shown in [**3**, Section 12.2] that we have the parity property

$$P_n(-x) = (-1)^n P_n(x) \quad \text{for } n \in \mathbb{N}_0 \text{ and } x \in [-1,1]\ , \tag{2.3}$$

and that, for all $n \in \mathbb{N}_0$, the Legendre polynomial P_n satisfies the Legendre differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-x^2\right)\frac{d}{\mathrm{d}x}\right)P_n(x) + n(n+1)P_n(x) = 0\ . \tag{2.4}$$

The latter could also be used to define the Legendre polynomials by letting P_n be the polynomial solution of the Legendre differential equation (2.4). As detailed in Lebedev [**17**, Section 4.5], the orthogonality of the Legendre polynomials

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = 0 \quad \text{for } n,m \in \mathbb{N}_0 \text{ with } n \neq m$$

follows from (2.4) and is applied together with the Bonnet recursion formula (2.1) to prove that

$$\int_{-1}^{1} \left(P_n(x)\right)^2 \mathrm{d}x = \frac{2}{2n+1} \quad \text{for } n \in \mathbb{N}_0\ . \tag{2.5}$$

Alternatively, Legendre polynomials could be defined as the sequence of polynomials orthogonal with respect to the weighting function 1 over $[-1,1]$ subject to requiring $P_n(1) = 1$ for all $n \in \mathbb{N}_0$, see Andrews, Askey and Roy [**2**, Remark 5.3.1]. The Legendre polynomials then arise by applying the Gram–Schmidt orthogonalisation process to the monomials $\{x^n : n \in \mathbb{N}_0\}$ on $[-1,1]$ with respect to the usual L^2 inner product and by imposing the normalisation $P_n(1) = 1$ for all $n \in \mathbb{N}_0$. With this approach the completeness of the Legendre polynomials follows immediately. Another option is to rewrite the Legendre differential equation (2.4) as an eigenvalue problem and to appeal to Sturm–Liouville theory, cf. [**3**, Chapter 10].

In our expressions for the iterated Kolmogorov loops, we need the family $\{Q_n : n \in \mathbb{N}_0\}$ of the shifted Legendre polynomials on the interval [0,1], which are given by

$$Q_n(t) = P_n(2t-1) \quad \text{for } t \in [0,1]\ .$$

These polynomials inherit their properties from the Legendre polynomials on $[-1,1]$. In particular, the shifted Legendre polynomials form a complete orthogonal system with

$$\int_{0}^{1} \left(Q_n(t)\right)^2 \mathrm{d}t = \frac{1}{2n+1} \quad \text{for } n \in \mathbb{N}_0\ , \tag{2.6}$$

and they satisfy the parity relation

$$Q_n(1-t) = (-1)^n Q_n(t) \quad \text{for } n \in \mathbb{N}_0 \text{ and } t \in [0,1] \; . \tag{2.7}$$

2.1. *Complex-valued Legendre polynomials*

We introduce a family $\{P_n : n \in \mathbb{Z}\}$ indexed by the integers $\mathbb{Z}$ of complex-valued polynomials on $[-1,1]$ which extends the family $\{P_n : n \in \mathbb{N}_0\}$ of Legendre polynomials on $[-1,1]$. When generalising the Legendre polynomials and dealing with associated Legendre polynomials, it is common to define the associated Legendre polynomial of negative degree $-n-1$ and zeroth order to equal P_n for $n \in \mathbb{N}_0$. The reason for this is that the Legendre differential equation (2.4) is invariant under a change from n to $-n-1$. However, we instead choose to set

$$P_{-n-1}(x) = \mathrm{i}\, P_n(x) \quad \text{for } n \in \mathbb{N}_0 \text{ and } x \in [-1,1] \; . \tag{2.8}$$

Our motivation for this choice is that, according to (2.5), it gives rise to

$$\int_{-1}^{1} (P_{-n-1}(x))^2 \, \mathrm{d}x = -\int_{-1}^{1} (P_n(x))^2 \, \mathrm{d}x = -\frac{2}{2n+1} = \frac{2}{2(-n-1)+1} \quad \text{for } n \in \mathbb{N}_0 \; ,$$

and therefore, we have

$$\int_{-1}^{1} (P_n(x))^2 \, \mathrm{d}x = \frac{2}{2n+1} \quad \text{for all } n \in \mathbb{Z} \; . \tag{2.9}$$

Moreover, the Bonnet recursion formula extends consistently across the original boundary at $n = 0$ to all $n \in \mathbb{Z}$.

LEMMA 2.1. *For all $n \in \mathbb{Z}$ and all $x \in [-1,1]$, we have*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \; .$$

Proof. For $n \in \mathbb{N}$, this is the usual Bonnet recursion formula (2.1). If $n \in \mathbb{Z} \setminus \mathbb{N}_0$ then, due to (2.8), we have

$$P_n(x) = \mathrm{i}\, P_{-n-1}(x) \quad \text{for } x \in [-1,1] \; ,$$

and we use (2.1) in the form

$$-nP_{-n}(x) = (-2n-1)xP_{-n-1}(x) - (-n-1)P_{-n-2}(x)$$

to deduce that, for all $x \in [-1,1]$,

$$\begin{aligned}(n+1)P_{n+1}(x) = (n+1)\,\mathrm{i}\, P_{-n-2}(x) &= (2n+1)x\,\mathrm{i}\, P_{-n-1}(x) - n\,\mathrm{i}\, P_{-n}(x) \\ &= (2n+1)xP_n(x) - nP_{n-1}(x) \; ,\end{aligned}$$

as required. For $n = 0$, we explicitly see that $P_1(x) = x$ coincides with $xP_0(x) = x$. □

This extension of the Legendre polynomials turns out to be convenient for our analysis. In the next section, we use these polynomials to introduce a family of complex-valued polynomials related to the integrals of the Legendre polynomials.

2.2. *Integrals of Legendre polynomials*

Let $\{I_n : n \in \mathbb{Z}\}$ be the family index by $\mathbb{Z}$ defined by

$$(2n+1)I_n(x) = P_{n+1}(x) - P_{n-1}(x) \quad \text{for } n \in \mathbb{Z} \text{ and } x \in [-1,1] \; . \tag{2.10}$$

The property (2.2) implies

$$I_n(x) = \int_{-1}^{x} P_n(z)\,\mathrm{d}z \quad \text{for all } n \in \mathbb{N} \text{ and } x \in [-1,1]\,. \tag{2.11}$$

However, we note that this relation does not hold for $n = 0$ because

$$I_0(x) = P_1(x) - P_{-1}(x) = P_1(x) - \mathrm{i}\,P_0(x) = x - \mathrm{i}\,,$$

whereas $\int_{-1}^{x} P_0(z)\,\mathrm{d}z = 1 + x$. This discrepancy is exploited to present a short proof of Lemma 4.7.

The parity property (2.3) yields

$$I_n(-x) = (-1)^{n+1} I_n(x) \quad \text{for } n \in \mathbb{N} \text{ and } x \in [-1,1]\,, \tag{2.12}$$

and in particular,

$$I_n(1) = I_n(-1) = 0 \quad \text{for all } n \in \mathbb{N}\,. \tag{2.13}$$

We further obtain the symmetry relation stated below as well as a recursion formula.

Lemma 2.2. *For all $n \in \mathbb{N}$ and all $x \in [-1,1]$, we have*

$$I_{-n-1}(x) = \mathrm{i}\,I_n(x)\,.$$

Proof. If $n \in \mathbb{N}$, then $n - 1 \in \mathbb{N}_0$ and therefore, by the definition (2.10) and by (2.8), we see

$$\begin{aligned} -(2n+1)I_{-n-1}(x) = (2(-n-1)+1)I_{-n-1}(x) &= P_{-n}(x) - P_{-n-2}(x) \\ &= \mathrm{i}\,P_{n-1}(x) - \mathrm{i}\,P_{n+1}(x) = -(2n+1)\,\mathrm{i}\,I_n(x)\,, \end{aligned}$$

which implies the desired result. □

In Lemma 2.2, it is important to restrict our attention to $n \in \mathbb{N}$ since for $n = 0$, we have

$$I_{-1}(x) = \mathrm{i}\,x - 1 \quad \text{and} \quad \mathrm{i}\,I_0(x) = \mathrm{i}\,x + 1\,.$$

Lemma 2.3. *For all $n \in \mathbb{Z}$ and all $x \in [-1,1]$, we have the recursion formula*

$$(n+2)I_{n+1}(x) = (2n+1)xI_n(x) - (n-1)I_{n-1}(x)\,.$$

Proof. This is a consequence of (2.10) and the extended Bonnet recursion formula, cf. Lemma 2.1. From

$$(n+2)P_{n+2}(x) = (2n+3)xP_{n+1}(x) - (n+1)P_n(x)\,,$$

we deduce

$$(n+2)I_{n+1}(x) = \frac{(n+2)(P_{n+2}(x) - P_n(x))}{2n+3} = xP_{n+1}(x) - P_n(x)\,, \tag{2.14}$$

and similarly,

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

implies

$$(n-1)I_{n-1}(x) = \frac{(n-1)(P_n(x) - P_{n-2}(x))}{2n-1} = P_n(x) - xP_{n-1}(x)\,. \tag{2.15}$$

Adding equation (2.14) to equation (2.15) yields

$$(n+2)I_{n+1}(x) + (n-1)I_{n-1}(x) = x(P_{n+1}(x) - P_{n-1}(x))\,,$$

and therefore, by (2.10),

$$(n+2)I_{n+1}(x) + (n-1)I_{n-1}(x) = (2n+1)xI_n(x)\ ,$$

as claimed. □

Throughout the moment analysis presented in Section 4, it is crucial, for example, see Lemma 4.2, that the above recursion formula holds for all $n \in \mathbb{Z}$ and that the original boundary case at $n = 0$ does not need a special treatment. For the latter, the discrepancy between I_0 and the integral of P_0 is also essential.

2.3. *Asymptotic behaviour*

We characterise the asymptotics in the limit $n \to \infty$ for Legendre polynomials and their integrals by relating these polynomials to Jacobi polynomials on $[-1, 1]$ and then quoting the Darboux formula for Jacobi polynomials.

Following Szegő [24, Section 4.22] and using the rising Pochhammer symbol, we define the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree $n \in \mathbb{N}_0$ on $[-1, 1]$ for $\alpha, \beta \in \mathbb{R}$ by

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+\alpha+\beta+1)_k(\alpha+k+1)_{n-k}\left(\frac{x-1}{2}\right)^k \quad \text{for } x \in [-1,1]\ . \tag{2.16}$$

If $\alpha, \beta > -1$, this agrees with the usual definition, cf. [2, Definition 2.5.1],

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \quad \text{for } x \in [-1,1]\ ,$$

where ${}_2F_1$ is the Gaussian hypergeometric function represented by the power series

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \quad \text{for } z \in (-1,1)\ .$$

For $\alpha, \beta > -1$ fixed, the polynomials $P_n^{(\alpha,\beta)}$ are orthogonal on $[-1, 1]$ with respect to the weighting function $(1-x)^\alpha(1+x)^\beta$. As discussed in [24, Section 4.21], the expression (2.16) implies

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \quad \text{for } n \in \mathbb{N} \text{ and } x \in [-1,1]\ . \tag{2.17}$$

As remarked in [24, Section 4.1], we further have

$$P_n^{(0,0)}(x) = P_n(x) \quad \text{for } n \in \mathbb{N}_0 \text{ and } x \in [-1,1]\ , \tag{2.18}$$

and from (2.11) as well as (2.17), it follows that

$$P_{n+1}^{(-1,-1)}(x) = \frac{1}{2}nI_n(x) \quad \text{for } n \in \mathbb{N} \text{ and } x \in [-1,1]\ . \tag{2.19}$$

An alternative derivation of (2.19) which uses the second-order differential equations satisfied by Jacobi polynomials is given by Belinsky [5]. Moreover, according to [5, Theorem 3], the polynomials $\{I_n : n \in \mathbb{N}\}$ are orthogonal on the interval $[-1, 1]$ with respect to the weighting function $(1-x^2)^{-1}$. However, as this weighting function is not continuous on $[-1, 1]$ these polynomials do not belong to the class of classical orthogonal polynomials.

To gain control over the Legendre polynomials and their integrals in the limit $n \to \infty$, we exploit an asymptotic property of Jacobi polynomials, cf. [24, Theorem 8.21.8], which is due to Darboux [8].

THEOREM 2.4 (Darboux formula). *Let $\alpha, \beta \in \mathbb{R}$ be arbitrary. For $\theta \in (0, \pi)$, set*

$$k(\theta) = \pi^{-\frac{1}{2}} \left(\sin \frac{\theta}{2} \right)^{-\alpha - \frac{1}{2}} \left(\cos \frac{\theta}{2} \right)^{-\beta - \frac{1}{2}} .$$

Then, as $n \to \infty$, we have

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-\frac{1}{2}} k(\theta) \cos\left(\left(n + \frac{\alpha+\beta+1}{2} \right)\theta - \left(\alpha + \frac{1}{2} \right)\frac{\pi}{2} \right) + O\left(n^{-\frac{3}{2}}\right) ,$$

where the bound on the error term holds uniformly in $\theta \in [\varepsilon, \pi - \varepsilon]$ for $\varepsilon > 0$.

By the Darboux formula, we particularly have, as $n \to \infty$,

$$P_n^{(0,0)}(\cos\theta) = \sqrt{\frac{2}{n\pi \sin\theta}} \cos\left(\left(n + \frac{1}{2} \right)\theta - \frac{\pi}{4} \right) + O\left(n^{-\frac{3}{2}}\right) , \quad \text{and} \tag{2.20}$$

$$P_n^{(-1,-1)}(\cos\theta) = \sqrt{\frac{\sin\theta}{2n\pi}} \cos\left(\left(n - \frac{1}{2} \right)\theta + \frac{\pi}{4} \right) + O\left(n^{-\frac{3}{2}}\right) , \tag{2.21}$$

uniformly in $\theta \in [\varepsilon, \pi - \varepsilon]$ for $\varepsilon > 0$. These asymptotics are used for estimates in Section 5.

3. *Iterated Kolmogorov loops*

We find two alternative representations for the iterated Kolmogorov loop of step N and we use the second representation, cf. Proposition 1.3, to prove Theorem 1.4. Although the first representation is obtained by applying the most evident approach of considering the first component of an expression for the iterated Kolmogorov diffusion $(\boldsymbol{B}_t^N)_{t\in[0,1]}$ of step N conditioned on $\boldsymbol{B}_1^N = 0$, the second representation in terms of shifted Legendre polynomials is much more useful for our analysis. This is due to the orthogonality of the Legendre polynomials. Moreover, we see that the first representation requires the inversion of a particular $N \times N$ factorial Hankel matrix.

Throughout, for $l \in \{1, \dots, N\}$ and $t \in [0, 1]$, we write $\boldsymbol{B}_t^{N,l}$ to denote the lth component of $\boldsymbol{B}_t^N$. We observe that integration by parts yields

$$\boldsymbol{B}_t^{N,l} = \int_0^t \int_0^{s_{l-1}} \dots \int_0^{s_2} B_{s_1} \, \mathrm{d}s_1 \dots \mathrm{d}s_{l-1} = \frac{1}{(l-1)!} \int_0^t (t-s)^{l-1} \, \mathrm{d}B_s . \tag{3.1}$$

To obtain the first representation for the iterated Kolmogorov loop of step N, we follow a similar line of reasoning as in [**12**, Section 4.4].

PROPOSITION 3.1. *Fix $N \in \mathbb{N}$. Let $\alpha_1, \dots, \alpha_N$ be the polynomials on* [0,1] *given by, for $t \in [0, 1]$,*

$$\alpha_l(t) = \sum_{k=1}^{N} (-1)^{N+k+l+1} (l-1)! \binom{N}{k} \binom{N+l-1}{l-1} \sum_{m=0}^{k-1} \binom{N-k+m}{l-1} \binom{N+m-1}{m} t^k .$$

Then the stochastic process $(Z_t^N)_{t\in[0,1]}$ in $\mathbb{R}$ defined by

$$Z_t^N = B_t - \sum_{l=1}^{N} \alpha_l(t) \boldsymbol{B}_1^{N,l}$$

has the same law as the iterated Kolmogorov loop of step N.

Proof. Let A be the $N \times N$ matrix and let E be the $N \times 1$ matrix which have the entries, for $k, l \in \{1, \dots, N\}$,

$$A_{kl} = \begin{cases} 1 & \text{if } k = l+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad E_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Using the matrix exponential of a square matrix, we set, for $r \in [0,1]$,

$$U(r) = \mathrm{e}^{rA} E \,, \tag{3.2}$$

and we define, for $t \in [0,1]$,

$$V(t) = \int_0^t U(t-s) U(-s)^T \, \mathrm{d}s \,. \tag{3.3}$$

Since

$$(U(r))_k = \frac{r^{k-1}}{(k-1)!} \,, \tag{3.4}$$

we compute with the help of [**12**, Lemma 3.2] that

$$(V(t))_{kl} = \frac{1}{(k-1)!\,(l-1)!} \int_0^t (t-s)^{k-1} (-s)^{l-1} \, \mathrm{d}s = (-1)^{l-1} \frac{t^{k+l-1}}{(k+l-1)!} \,.$$

We further observe that due to (3.1) and (3.4) the iterated Kolmogorov diffusion of step N can be expressed as

$$\boldsymbol{B}_t^N = \int_0^t U(t-s) \, \mathrm{d}B_s \,. \tag{3.5}$$

Let $(\boldsymbol{Z}_t^N)_{t \in [0,1]}$ be the stochastic process in $\mathbb{R}^N$ given by

$$\boldsymbol{B}_t^N = \boldsymbol{Z}_t^N + V(t) V(1)^{-1} \boldsymbol{B}_1^N \,. \tag{3.6}$$

Using the expression (3.5), applying the Itô isometry and recalling the definitions (3.2) and (3.3), we obtain

$$\mathbb{E}\Big[\boldsymbol{B}_t^N \big(\boldsymbol{B}_1^N\big)^T \Big] = \int_0^t U(t-s) U(1-s)^T \, \mathrm{d}s = V(t) \big(\mathrm{e}^A\big)^T \,.$$

It follows that

$$\mathbb{E}\Big[\boldsymbol{Z}_t^N \big(\boldsymbol{B}_1^N\big)^T \Big] = V(t) \big(\mathrm{e}^A\big)^T - V(t) V(1)^{-1} V(1) \big(\mathrm{e}^A\big)^T = 0 \quad \text{for all } t \in [0,1] \,.$$

Since both the process $(\boldsymbol{Z}_t^N)_{t \in [0,1]}$ and the random variable $\boldsymbol{B}_1^N$ have zero mean and are Gaussian, they are uncorrelated which implies that they are independent. Therefore, we deduce from (3.6) that $(\boldsymbol{Z}_t^N)_{t \in [0,1]}$ is the process obtained by conditioning $(\boldsymbol{B}_t^N)_{t \in [0,1]}$ on $\boldsymbol{B}_1^N = 0$, and it suffices to show that the first component of $(\boldsymbol{Z}_t^N)_{t \in [0,1]}$ is $(Z_t^N)_{t \in [0,1]}$. This requires an explicit expression for the inverse $(V(1))^{-1}$, which is easily derived from the formula given in [**10**]. We have

$$\big(V(1)^{-1}\big)_{kl} = (-1)^{N+l} (k-1)!\, l! \binom{N-1}{k-1} \binom{N+l-1}{l} \sum_{m=0}^{k-1} \binom{N-k+m}{l-1} \binom{N+m-1}{m} \,,$$

and hence, for all $l \in \{1, \dots, N\}$ and all $t \in [0,1]$, we see that

$$\big(V(t) V(1)^{-1}\big)_{1l} = \sum_{k=1}^N (V(t))_{1k} \big(V(1)^{-1}\big)_{kl} = \alpha_l(t) \,.$$

Thus, we conclude

$$\boldsymbol{Z}_t^{N,1} = \boldsymbol{B}_t^{N,1} - \sum_{l=1}^{N} \left(V(t)V(1)^{-1}\right)_{1l} \boldsymbol{B}_1^{N,l} = B_t - \sum_{l=1}^{N} \alpha_l(t) \boldsymbol{B}_1^{N,l} ,$$

as needed. □

While the representation given in Proposition 3.1 is obtained through a straightforward approach, it appears to be too complicated to proceed with, amongst others because the polynomial coefficients $\alpha_1, \dots, \alpha_N$ in front of the components of $\boldsymbol{B}_1^N$ change with N. Instead, we use the representation given in Proposition 1.3 for our analysis.

Proof of Proposition 1.3. Since the shifted Legendre polynomial Q_n is a polynomial of degree n which satisfies the parity relation (2.7), it follows from (3.1) that $\int_0^1 Q_n(r)\,\mathrm{d}B_r$ can be expressed as a linear combination of

$$B_1, \int_0^1 B_{s_1}\,\mathrm{d}s_1, \int_0^1 \int_0^{s_2} B_{s_1}\,\mathrm{d}s_1\,\mathrm{d}s_2, \dots, \int_0^1 \int_0^{s_n} \dots \int_0^{s_2} B_{s_1}\,\mathrm{d}s_1 \dots \mathrm{d}s_n .$$

Thus, for $N \in \mathbb{N}$ fixed, there exist polynomials $\beta_1, \dots, \beta_N$ on [0,1] such that, for $t \in [0,1]$,

$$\sum_{n=0}^{N-1} (2n+1) \int_0^t Q_n(r)\,\mathrm{d}r \int_0^1 Q_n(r)\,\mathrm{d}B_r = \sum_{l=1}^{N} \beta_l(t) \boldsymbol{B}_1^{N,l} .$$

As the process $(L_t^N)_{t\in[0,1]}$ is defined, according to (1.1), by

$$L_t^N = B_t - \sum_{n=0}^{N-1} (2n+1) \int_0^t Q_n(r)\,\mathrm{d}r \int_0^1 Q_n(r)\,\mathrm{d}B_r ,$$

we can write

$$B_t = L_t^N + \sum_{l=1}^{N} \beta_l(t) \boldsymbol{B}_1^{N,l} . \tag{3.7}$$

Using the Itô isometry and the orthogonality of the shifted Legendre polynomials with (2.6), we obtain from (1.1) that, for all $t \in [0,1]$ and all $m \in \{0, \dots, N-1\}$,

$$\mathbb{E}\left[L_t^N \int_0^1 Q_m(r)\,\mathrm{d}B_r\right] = \int_0^t Q_m(r)\,\mathrm{d}r - \sum_{n=0}^{N-1} (2n+1) \int_0^t Q_n(r)\,\mathrm{d}r \int_0^1 Q_n(r) Q_m(r)\,\mathrm{d}r = 0 .$$

By the completeness of the shifted Legendre polynomials and the identity (3.1), this implies that, for all $t \in [0,1]$ and all $l \in \{1, \dots, N\}$,

$$\mathbb{E}\left[L_t^N \boldsymbol{B}_1^{N,l}\right] = 0 .$$

Hence, L_t^N and $\boldsymbol{B}_1^N$ are uncorrelated for all $t \in [0,1]$, which due to $(L_t^N)_{t\in[0,1]}$ and $\boldsymbol{B}_1^N$ both being Gaussian shows that $(L_t^N)_{t\in[0,1]}$ and $\boldsymbol{B}_1^N$ are independent. From the representation (3.7), we finally deduce that $(L_t^N)_{t\in[0,1]}$ is indeed equal in law to the iterated Kolmogorov loop of step N. □

The advantage of the representation for the iterated Kolmogorov loops given in Proposition 1.3 over the one given in Proposition 3.1 is that by the orthogonality of the shifted Legendre polynomials, it gives rise to a neat expression for the covariance functions of the iterated Kolmogorov loops.

LEMMA 3.2. *The iterated Kolmogorov loop of step $N \in \mathbb{N}$ is a zero-mean Gaussian process with covariance C_N given, for $s, t \in [0,1]$, by*

$$C_N(s,t) = \min(s,t) - \sum_{n=0}^{N-1} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^t Q_n(r)\,\mathrm{d}r\ .$$

Proof. By Proposition 1.3, it suffices to show that $(L_t^N)_{t\in[0,1]}$ is a zero-mean Gaussian process with the specified covariance function. From the definition (1.1), we see that $(L_t^N)_{t\in[0,1]}$ is a zero-mean Gaussian process. Regarding its covariance function, the Itô isometry and the orthogonality of the shifted Legendre polynomials with (2.6) imply that, for $s, t \in [0,1]$,

$$\mathbb{E}\left[B_s \sum_{n=0}^{N-1} (2n+1) \int_0^t Q_n(r)\,\mathrm{d}r \int_0^1 Q_n(r)\,\mathrm{d}B_r\right] = \sum_{n=0}^{N-1} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^t Q_n(r)\,\mathrm{d}r$$

as well as

$$\begin{aligned}&\mathbb{E}\left[\left(\sum_{n=0}^{N-1} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^1 Q_n(r)\,\mathrm{d}B_r\right)\left(\sum_{n=0}^{N-1} (2n+1) \int_0^t Q_n(r)\,\mathrm{d}r \int_0^1 Q_n(r)\,\mathrm{d}B_r\right)\right]\\ &\quad= \sum_{n=0}^{N-1} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^t Q_n(r)\,\mathrm{d}r\ ,\end{aligned}$$

which together with $\mathbb{E}[B_sB_t] = \min(s,t)$ yields

$$C_N(s,t) = \mathbb{E}\big[L_s^N L_t^N\big] = \min(s,t) - \sum_{n=0}^{N-1} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^t Q_n(r)\,\mathrm{d}r\ ,$$

as required. □

This characterisation of the iterated Kolmogorov loops allows us to prove Theorem 1.4 by exploiting the completeness and orthogonality of the shifted Legendre polynomials. The argument follows a line of reasoning which is part of the usual proof of Mercer's theorem, see [**19**, Part IV].

Proof of Theorem 1.4. From the discussion in Section 2, we recall $\{\sqrt{2n+1}Q_n : n \in \mathbb{N}_0\}$ forms a complete orthonormal set of polynomials in $L^2[0,1]$ with respect to the usual inner product. As a consequence, the polarised Parseval identity applies to give, for $s, t \in [0,1]$,

$$\begin{aligned}\min(s,t) = \int_0^1 \mathbb{1}_{[0,s]}(r)\mathbb{1}_{[0,t]}(r)\,\mathrm{d}r &= \sum_{n=0}^{\infty} (2n+1) \int_0^1 \mathbb{1}_{[0,s]}(r) Q_n(r)\,\mathrm{d}r \int_0^1 \mathbb{1}_{[0,t]}(r) Q_n(r)\,\mathrm{d}r\\ &= \sum_{n=0}^{\infty} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^t Q_n(r)\,\mathrm{d}r.\end{aligned} \tag{3.8}$$

Due to Lemma 3.2, it follows that the covariance C_N of the iterated Kolmogorov loop of step N is given, for $s, t \in [0,1]$, by

$$C_N(s,t) = \sum_{n=N}^{\infty} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^t Q_n(r)\,\mathrm{d}r\ .$$

Using Cauchy–Schwarz, we obtain that, for all $N, M \in \mathbb{N}$ with $N < M$ and for $s, t \in [0,1]$ fixed,

$$\left|\sum_{n=N}^{M}(2n+1)\left|\int_0^s Q_n(r)\,\mathrm{d}r\right|\left|\int_0^t Q_n(r)\,\mathrm{d}r\right|\right|^2 \leqslant \sum_{n=N}^{M}(2n+1)\left(\int_0^s Q_n(r)\,\mathrm{d}r\right)^2 \sum_{n=N}^{M}(2n+1)\left(\int_0^t Q_n(r)\,\mathrm{d}r\right)^2,$$

and thus, by (3.8), we have

$$\sum_{n=N}^{M}(2n+1)\left|\int_0^s Q_n(r)\,\mathrm{d}r\right|\left|\int_0^t Q_n(r)\,\mathrm{d}r\right| \leqslant \sqrt{st}\,.$$

This implies that the series representation for $\min(s,t)$ in (3.8) converges absolutely. In particular, the sequence $(G_N)_{N\in\mathbb{N}}$ of functions $G_N\colon [0,1]\times[0,1]\to\mathbb{R}$ defined by

$$G_N(s,t) = \sum_{n=N}^{\infty}(2n+1)\left|\int_0^s Q_n(r)\,\mathrm{d}r\right|\left|\int_0^t Q_n(r)\,\mathrm{d}r\right|$$

converges pointwise to zero as $N\to\infty$. As $(G_N)_{N\in\mathbb{N}}$ is a monotonically decreasing sequence of continuous real-valued functions, Dini's theorem applies to give that $(G_N)_{N\in\mathbb{N}}$ converges uniformly on the compact set $[0,1]\times[0,1]$ to the zero function. By the Cauchy criterion, we further deduce that the sequence $(C_N)_{N\in\mathbb{N}}$ of covariances converges uniformly on $[0,1]\times[0,1]$ to the zero function. As the iterated Kolmogorov loops are zero-mean Gaussian processes and since their covariance functions converge uniformly as $N\to\infty$ to zero, it follows, for example, by [**15**, Section 3], that the iterated Kolmogorov loops of step N indeed converge weakly as $N\to\infty$ to the zero process on $\Omega^{0,0}$. □

Note that Proposition 1.3 and Theorem 1.4 together show that Brownian motion $(B_t)_{t\in[0,1]}$ admits the decomposition

$$\left(\sum_{n=0}^{\infty}(2n+1)\int_0^t Q_n(r)\,\mathrm{d}r\int_0^1 Q_n(r)\,\mathrm{d}B_r\right)_{t\in[0,1]},$$

which differs from the usual Karhunen–Loève expansion, cf. [**18**, p. 144], for Brownian motion, and which alternatively could be expressed in terms of the representation given in Proposition 3.1. Foster, Lyons and Oberhauser [**9**] independently obtained this decomposition with the difference that the random coefficients of the integrals of the shifted Legendre polynomials are defined using the Brownian bridge process associated with $(B_t)_{t\in[0,1]}$. They use this representation to generate approximate sample paths of Brownian motion which respect integration of polynomials up to a fixed degree.

Based on our works, Trefethen implemented the resulting polynomial approximation of Brownian motion numerically as a Chebfun Example, see [**25**].

4. *Moment analysis on the diagonal*

As the first step towards proving Theorem 1.5, we establish the convergence of moments on the diagonal. Throughout, we use the families of complex-valued polynomials introduced in Sections 2.1 and 2.2 to simplify the presentation of our analysis. We repeatedly expand terms into their partial fraction decomposition because this reveals that certain sums we encounter telescope. Let $S_N\colon[-1,1]\to\mathbb{R}$ be the restriction of R_N to the diagonal, that is,

$$S_N(x)=R_N(x,x)\quad\text{for } x\in[-1,1]\,.$$

Due to (2.11), we can write

$$S_N(x)=N\left(1+x-\frac{1}{2}\left((1+x)^2+\sum_{n=1}^{N-1}(2n+1)\left(I_n(x)\right)^2\right)\right)\quad\text{for } x\in[-1,1]\,.$$

To study the moments of S_N in the limit $N\to\infty$, we start by considering each summand separately. In particular, for all $k\in\mathbb{N}_0$, we have

$$\int_{-1}^{1}x^{2k}(1+x)\,\mathrm{d}x=\frac{2}{2k+1}\quad\text{and}\quad\int_{-1}^{1}x^{2k+1}(1+x)\,\mathrm{d}x=\frac{2}{2k+3}\tag{4.1}$$

as well as

$$\frac{1}{2}\int_{-1}^{1}x^{2k}(1+x)^2\,\mathrm{d}x=\frac{1}{2k+1}+\frac{1}{2k+3}\quad\text{and}\quad\frac{1}{2}\int_{-1}^{1}x^{2k+1}(1+x)^2\,\mathrm{d}x=\frac{2}{2k+3}.\tag{4.2}$$

The remaining odd moments all vanish.

LEMMA 4.1. *For all $n\in\mathbb{N}$ and all $k\in\mathbb{N}_0$, we have*

$$\int_{-1}^{1}x^{2k+1}\left(I_n(x)\right)^2\,\mathrm{d}x=0\,.$$

Proof. By the parity property (2.12), we know that I_n^2 is an even function on $[-1,1]$ for all $n\in\mathbb{N}$. Therefore, the integrand of the above integral is an odd function on $[-1,1]$, and it follows that the integral vanishes. □

We are left with studying the remaining even moments, which is the core of our moment analysis. The recursive method we develop requires us to look at additional moments to the ones we would like to consider. For all $p,q\in\mathbb{Z}$ and $k\in\mathbb{N}_0$, we set

$$m_{p,q}^k=(p+q+1)\int_{-1}^{1}x^{2k}I_p(x)I_q(x)\,\mathrm{d}x\,.$$

These moments satisfy the following recursion formula. This is the first time where the extension of the family of the integrals of the Legendre polynomials comes in handy as we do not have to deal with a boundary at $n=0$.

LEMMA 4.2. *For all $p,q\in\mathbb{Z}$ and for $k\in\mathbb{N}$, we have*

$$\begin{aligned}m_{p,q}^k=&\frac{(p+q+1)(p+2)(q+2)}{(2p+1)(2q+1)(p+q+3)}m_{p+1,q+1}^{k-1}+\frac{(p+q+1)(p-1)(q-1)}{(2p+1)(2q+1)(p+q-1)}m_{p-1,q-1}^{k-1}\\&+\frac{(p+2)(q-1)}{(2p+1)(2q+1)}m_{p+1,q-1}^{k-1}+\frac{(p-1)(q+2)}{(2p+1)(2q+1)}m_{p-1,q+1}^{k-1}\,.\end{aligned}$$

Proof. According to Lemma 2.3, we have both

$$(p+2)I_{p+1}(x)=(2p+1)xI_p(x)-(p-1)I_{p-1}(x)\quad\text{for } x\in[-1,1]$$

and

$$(q+2)I_{q+1}(x)=(2q+1)xI_q(x)-(q-1)I_{q-1}(x)\quad\text{for } x\in[-1,1].$$

It follows that

$$(2p+1)(2q+1)\int_{-1}^{1} x^{2k} I_p(x) I_q(x)\,\mathrm{d}x$$
$$= \int_{-1}^{1} x^{2k-2}((p+2)I_{p+1}(x) + (p-1)I_{p-1}(x))((q+2)I_{q+1}(x) + (q-1)I_{q-1}(x))\,\mathrm{d}x\ ,$$

which yields the desired result. □

Moreover, we have the partial fraction decompositions specified below.

PROPOSITION 4.3. *There exists a family* $\{b^l_{a,k} \in \mathbb{R} \colon a \in \mathbb{Z}$ *and* $k, l \in \mathbb{N}_0\}$ *of coefficients satisfying*

$$b^l_{a,k} = b^l_{-a,k} \quad \text{for all } a \in \mathbb{Z} \text{ and } k, l \in \mathbb{N}_0 \tag{4.3}$$

as well as

$$b^{a-1}_{0,k} + 2a \sum_{l=0}^{k} \frac{b^l_{a,k}}{l+1} = 0 \quad \text{and} \quad \frac{b^{a-1}_{c,k}}{a} = \frac{b^{c-1}_{a,k}}{c} \quad \text{for all } a, c \in \mathbb{N} \text{ and } k \in \mathbb{N}_0\ , \tag{4.4}$$

such that, for all $n, a \in \mathbb{Z}$ *and all* $k \in \mathbb{N}_0$,

$$m^k_{n-a,n+a} = \sum_{l=0}^{k} \frac{b^l_{a,k}}{2n-2l-1} - \sum_{l=0}^{k} \frac{b^l_{a,k}}{2n+2l+3}\ , \tag{4.5}$$

where it is understood that $b^l_{a,k} = 0$ *if* $l > k$.

Proof. The proof works by induction on $k \in \mathbb{N}_0$. For the base case, we start by observing that the orthogonality of the Legendre polynomials and the definition (2.8) imply that

$$\int_{-1}^{1} P_n(x) P_m(x)\,\mathrm{d}x = 0 \quad \text{if } n \neq m \text{ and } n \neq -m-1\ . \tag{4.6}$$

In particular, the integral vanishes if $n \neq m$ but $n+m$ is even. Using (2.10) and (2.9), we compute that, for all $n \in \mathbb{Z}$,

$$m^0_{n,n} = (2n+1)\int_{-1}^{1} (I_n(x))^2\,\mathrm{d}x = \frac{1}{2n+1}\int_{-1}^{1} (P_{n+1}(x) - P_{n-1}(x))^2\,\mathrm{d}x$$
$$= \frac{1}{2n+1}\left(\frac{2}{2n-1} + \frac{2}{2n+3}\right) = \frac{1}{2n-1} - \frac{1}{2n+3}\ ,$$

and similarly,

$$m^0_{n-1,n+1} = m^0_{n+1,n-1} = (2n+1)\int_{-1}^{1} I_{n-1}(x) I_{n+1}(x)\,\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+3}\right)$$

as well as

$$m^0_{n-a,n+a} = 0 \quad \text{for all } a \in \mathbb{Z} \setminus \{-1, 0, 1\}\ .$$

Hence, for $k = 0$, the moments are indeed of the form (4.5) with the only non-zero coefficients

$$b^0_{0,0} = 1 \quad \text{and} \quad b^0_{1,0} = b^0_{-1,0} = -\frac{1}{2}\ . \tag{4.7}$$

In particular, we have $b^0_{0,0} + 2b^0_{1,0} = 0$ and the relations (4.3) as well as (4.4) are satisfied for $k = 0$, which settles the base case. For the induction step, we start with $a = 0$. Applying Lemma 4.2 and the induction hypothesis yields

$$\begin{aligned} m^k_{n,n} = {}& \frac{(n+2)^2}{(2n+1)(2n+3)}\left(\sum_{l=0}^{k-1}\frac{b^l_{0,k-1}}{2n-2l+1} - \sum_{l=0}^{k-1}\frac{b^l_{0,k-1}}{2n+2l+5}\right) \\ &+ \frac{(n-1)^2}{(2n-1)(2n+1)}\left(\sum_{l=0}^{k-1}\frac{b^l_{0,k-1}}{2n-2l-3} - \sum_{l=0}^{k-1}\frac{b^l_{0,k-1}}{2n+2l+1}\right) \\ &+ \frac{2(n-1)(n+2)}{(2n+1)^2}\left(\sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{2n-2l-1} - \sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{2n+2l+3}\right) . \end{aligned} \tag{4.8}$$

Through this expression, we can define what it means to evaluate $(2n+1)^2 m^k_{n,n}$ at $n = -1/2$. By additionally using the relation (4.4) of the induction hypothesis, we obtain that

$$(2n+1)^2 m^k_{n,n}\big|_{n=-1/2} = \frac{9}{4}b^0_{0,k-1} + \frac{9}{2}\sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{l+1} = \frac{9}{4}\left(b^0_{0,k-1} + 2\sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{l+1}\right) = 0 \,. \tag{4.9}$$

Since the remaining factors in the denominators of the terms giving $m^k_{n,n}$ only ever appear linearly, we deduce from (4.8) as well as (4.9), and by referring to the Heaviside cover-up method that $m^k_{n,n}$ is of the form

$$m^k_{n,n} = \sum_{l=0}^{k}\frac{b^l_{0,k}}{2n-2l-1} + \sum_{l=0}^{k}\frac{c^l_{0,k}}{2n+2l+3} + \frac{d_{0,k}}{2n+1} \,, \tag{4.10}$$

for suitable coefficients $b^l_{0,k}, c^l_{0,k}, d_{0,k} \in \mathbb{R}$. By Lemma 2.2, we further have

$$m^k_{-n-1,-n-1} = -(2n+1)\int_{-1}^{1} x^{2k}\left(I_{-n-1}(x)\right)^2 \mathrm{d}x = m^k_{n,n} \quad \text{for all } n \in \mathbb{N} \,. \tag{4.11}$$

However, according to (4.10), we can write

$$m^k_{-n-1,-n-1} = -\sum_{l=0}^{k}\frac{c^l_{0,k}}{2n-2l-1} - \sum_{l=0}^{k}\frac{b^l_{0,k}}{2n+2l+3} - \frac{d_{0,k}}{2n+1} \,,$$

and as a consequence of (4.11), it follows that

$$d_{0,k} = 0 \quad \text{and} \quad c^l_{0,k} = -b^l_{0,k} \quad \text{for } l \in \{0, \dots, k\} \,,$$

which gives the desired form (4.5) for $a = 0$. Similarly, for $a = \pm 1$, we use

$$\begin{aligned} m^k_{n-1,n+1} = m^k_{n+1,n-1} = {}& \frac{(2n+1)(n+1)(n+3)}{(2n-1)(2n+3)^2}\left(\sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{2n-2l+1} - \sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{2n+2l+5}\right) \\ &+ \frac{(2n+1)(n-2)n}{(2n-1)^2(2n+3)}\left(\sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{2n-2l-3} - \sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{2n+2l+1}\right) \\ &+ \frac{n(n+1)}{(2n-1)(2n+3)}\left(\sum_{l=0}^{k-1}\frac{b^l_{0,k-1}}{2n-2l-1} - \sum_{l=0}^{k-1}\frac{b^l_{0,k-1}}{2n+2l+3}\right) \\ &+ \frac{(n-2)(n+3)}{(2n-1)(2n+3)}\left(\sum_{l=0}^{k-1}\frac{b^l_{2,k-1}}{2n-2l-1} - \sum_{l=0}^{k-1}\frac{b^l_{2,k-1}}{2n+2l+3}\right) \end{aligned}$$

to give a meaning to

$$(2n-1)^2 m^k_{n-1,n+1}\big|_{n=1/2} = (2n+3)^2 m^k_{n-1,n+1}\big|_{n=-3/2}$$
$$= \frac{3}{16}\left(b^0_{0,k-1} + 2\sum_{l=0}^{k-1}\frac{b^l_{1,k-1}}{l+1}\right) + \frac{21}{16}\left(\frac{b^1_{1,k-1}}{2} - b^0_{2,k-1}\right),$$

whereas, for $a \in \mathbb{Z} \setminus \{-1,0,1\}$, we have

$$\begin{aligned} m^k_{n-a,n+a} = {} & \frac{(2n+1)(n-a+2)(n+a+2)}{(2n-2a+1)(2n+2a+1)(2n+3)}\left(\sum_{l=0}^{k-1}\frac{b^l_{a,k-1}}{2n-2l+1} - \sum_{l=0}^{k-1}\frac{b^l_{a,k-1}}{2n+2l+5}\right) \\ & + \frac{(2n+1)(n-a-1)(n+a-1)}{(2n-2a+1)(2n+2a+1)(2n-1)}\left(\sum_{l=0}^{k-1}\frac{b^l_{a,k-1}}{2n-2l-3} - \sum_{l=0}^{k-1}\frac{b^l_{a,k-1}}{2n+2l+1}\right) \\ & + \frac{(n-a+2)(n+a-1)}{(2n-2a+1)(2n+2a+1)}\left(\sum_{l=0}^{k-1}\frac{b^l_{a-1,k-1}}{2n-2l-1} - \sum_{l=0}^{k-1}\frac{b^l_{a-1,k-1}}{2n+2l+3}\right) \\ & + \frac{(n-a-1)(n+a+2)}{(2n-2a+1)(2n+2a+1)}\left(\sum_{l=0}^{k-1}\frac{b^l_{a+1,k-1}}{2n-2l-1} - \sum_{l=0}^{k-1}\frac{b^l_{a+1,k-1}}{2n+2l+3}\right) \end{aligned}$$

to make sense of

$$(2n-2a+1)^2 m^k_{n-a,n+a}\big|_{n=a-1/2} = (2n+2a+1)^2 m^k_{n-a,n+a}\big|_{n=-a-1/2}$$
$$= \frac{3(4a+3)}{16}\left(\frac{b^a_{a,k-1}}{a+1} - \frac{b^{a-1}_{a+1,k-1}}{a}\right) + \frac{3(4a-3)}{16}\left(\frac{b^{a-1}_{a-1,k-1}}{a} - \frac{b^{a-2}_{a,k-1}}{a-1}\right).$$

By the relation (4.4) of the induction hypothesis, it follows that, for all $a \in \mathbb{Z}$,

$$(2n-2a+1)^2 m^k_{n-a,n+a}\big|_{n=a-1/2} = (2n+2a+1)^2 m^k_{n-a,n+a}\big|_{n=-a-1/2} = 0\,. \tag{4.12}$$

Hence, as before, we use the expression for the moments, the Heaviside cover-up method and the symmetry property

$$m^k_{-n-1-a,-n-1+a} = -(2n+1)\int_{-1}^{1} x^{2k} I_{-(n+a)-1}(x) I_{-(n-a)-1}(x)\,\mathrm{d}x$$
$$= m^k_{n-a,n+a} \quad \text{for } n \geqslant a+1$$

to deduce that we indeed have the partial fraction decomposition (4.5). The symmetry relation (4.3) is satisfied since

$$m^k_{n-a,n+a} = m^k_{n+a,n-a} \quad \text{for all } n,a \in \mathbb{Z}\,.$$

To conclude the induction step, we still need to show that (4.4) holds. As we have just established that the moments are of the form (4.5), we are justified to define, for $a, c \in \mathbb{N}_0$,

$$d^c_{a,k} = \left.\frac{(2n-2c+1)m^k_{n-a,n+a}}{2n+1}\right|_{n=c-1/2},$$

where it is understood that

$$d^0_{a,k} = m^k_{n-a,n+a}\big|_{n=-1/2} = -\sum_{l=0}^{k}\frac{b^l_{a,k}}{l+1},$$

and where, for $c \in \mathbb{N}$, we have

$$d^c_{a,k} = \frac{b^{c-1}_{a,k}}{2c} \ .$$

Thus, the relation (4.4) of the induction hypothesis tells us that

$$d^a_{c,k-1} = d^c_{a,k-1} \quad \text{for all } a, c \in \mathbb{N}_0 \ . \tag{4.13}$$

Using Lemma 4.2, we obtain

$$\begin{aligned} d^a_{c,k} = {} & \frac{(2a-2c+3)(2a+2c+3)}{16(a-c)(a+c)} d^{a+1}_{c,k-1} + \frac{(2a-2c-3)(2a+2c-3)}{16(a-c)(a+c)} d^{a-1}_{c,k-1} \\ & + \frac{(2a-2c+3)(2a+2c-3)}{16(a-c)(a+c)} d^a_{c-1,k-1} + \frac{(2a-2c-3)(2a+2c+3)}{16(a-c)(a+c)} d^a_{c+1,k-1} \end{aligned} \tag{4.14}$$

as well as

$$\begin{aligned} d^c_{a,k} = {} & \frac{(2c-2a+3)(2c+2a+3)}{16(c-a)(c+a)} d^{c+1}_{a,k-1} + \frac{(2c-2a-3)(2c+2a-3)}{16(c-a)(c+a)} d^{c-1}_{a,k-1} \\ & + \frac{(2c-2a+3)(2c+2a-3)}{16(c-a)(c+a)} d^c_{a-1,k-1} + \frac{(2c-2a-3)(2c+2a+3)}{16(c-a)(c+a)} d^c_{a+1,k-1} \ . \end{aligned} \tag{4.15}$$

Due to (4.13), the first summand on the right-hand side of (4.14) agrees with the fourth summand on the right-hand side of (4.15). Similarly, the second summand in (4.14) coincides with the third summand in (4.15). As the remaining terms also match, we see that

$$d^a_{c,k} = d^c_{a,k} \quad \text{for all } a, c \in \mathbb{N}_0 \ ,$$

which implies the relation (4.4) and concludes the proof. □

By a more in-depth analysis than the one performed in the proof of Proposition 4.3, it is possible to use the Heaviside cover-up method to obtain recurrence relations for the coefficients $b^l_{a,k}$ which characterise them uniquely. However, as it is not necessary for our subsequent analysis to determine each coefficient $b^l_{a,k}$ separately, we postpone the derivation of recursion formulae to the Appendix. In the following, we see that to study the moments of S_N in the limit $N \to \infty$ it suffices to gain control over, for $a \in \mathbb{Z}$ and $k \in \mathbb{N}_0$,

$$B_{a,k} = \sum_{l=0}^{k} (l+1) b^l_{a,k} \ .$$

These sums satisfy a much simpler recurrence relation than the coefficients $b^l_{a,k}$ themselves, where $B_{a,k} = B_{-a,k}$ as a result of the symmetry property (4.3).

Proposition 4.4. *For all $k \in \mathbb{N}$ and all $a \in \mathbb{Z}$, we have*

$$B_{a,k} = \frac{1}{4} B_{a-1,k-1} + \frac{1}{2} B_{a,k-1} + \frac{1}{4} B_{a+1,k-1} \ . \tag{4.16}$$

Proof. Using the partial fraction decomposition (4.5) of Proposition 4.3 and

$$\frac{1}{2n-2l-1} - \frac{1}{2n+2l+3} = \frac{4(l+1)}{(2n-2l-1)(2n+2l+3)} \ ,$$

we deduce that

$$\lim_{n\to\infty} n^2 m^k_{n-a,n+a} = \lim_{n\to\infty} \sum_{l=0}^{k} \frac{4n^2(l+1)}{(2n-2l-1)(2n+2l+3)} b^l_{a,k} = B_{a,k} \ , \tag{4.17}$$

and similarly,

$$\lim_{n\to\infty}(n-1)^2 m^k_{n-a,n+a} = \lim_{n\to\infty}(n+1)^2 m^k_{n-a,n+a} = B_{a,k}\ . \tag{4.18}$$

On the other hand, by applying Lemma 4.2 with $p = n - a$ and $q = n + a$, we obtain that

$$\begin{aligned}\lim_{n\to\infty} n^2 m^k_{n-a,n+a} &= \frac{1}{4}\lim_{n\to\infty} n^2 m^{k-1}_{n-a+1,n+a+1} + \frac{1}{4}\lim_{n\to\infty} n^2 m^{k-1}_{n-a-1,n+a-1}\\ &\quad + \frac{1}{4}\lim_{n\to\infty} n^2 m^{k-1}_{n-a+1,n+a-1} + \frac{1}{4}\lim_{n\to\infty} n^2 m^{k-1}_{n-a-1,n+a+1}\ ,\end{aligned}$$

which together with (4.17) and (4.18) implies the claimed recurrence relation. □

Remark 4.5. The recurrence relation (4.16) in Proposition 4.4 can be rewritten as

$$4^k B_{a,k} = 4^{k-1}B_{a-1,k-1} + 2\big(4^{k-1}B_{a,k-1}\big) + 4^{k-1}B_{a+1,k-1} \quad \text{for } k\in\mathbb{N} \text{ and } a\in\mathbb{Z}\ ,$$

that is, the numbers $4^k B_{a,k}$ satisfy the same recurrence relation as elements of the Catalan triangle which Shapiro introduced in [**23**] and as elements of other Catalan triangles, for example, see [**20, 21**].

By the preceding remark, it should not come as a surprise that we encounter the Catalan numbers when determining the sums $B_{a,k}$ explicitly. For $k \in \mathbb{N}_0$, the kth Catalan number C_k is given by

$$C_k = \frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1}\ .$$

In the next lemma, it is understood that

$$\binom{k}{l} = 0 \quad \text{if } k,l\in\mathbb{N}_0 \text{ with } k<l\ .$$

Lemma 4.6. *We have $B_{0,0} = 1$ and, for $a, k \in \mathbb{N}_0$ with $a + k \geqslant 1$,*

$$B_{-a,k} = B_{a,k} = 4^{-k}\left[\binom{2k}{k+a} - \frac{1}{2}\left[\binom{2k}{k+a-1} + \binom{2k}{k+a+1}\right]\right]. \tag{4.19}$$

In particular, we see that

$$B_{0,k} = 4^{-k}C_k \quad \textit{for all } k\in\mathbb{N}_0\ . \tag{4.20}$$

Proof. By the recursion formula in Lemma 2.3 and the definition (2.10), the polynomial $x^k I_{n-a}(x)$ is a linear combination of the polynomials

$$P_{n-a-k-1}(x), P_{n-a-k+1}(x), \ldots, P_{n-a+k-1}(x), P_{n-a+k+1}(x)\ , \tag{4.21}$$

and similarly, $x^k I_{n+a}(x)$ is a linear combination of

$$P_{n+a-k-1}(x), P_{n+a-k+1}(x), \ldots, P_{n+a+k-1}(x), P_{n+a+k+1}(x)\ . \tag{4.22}$$

Hence, if $n - a + k + 1 < n + a - k - 1$, that is, if $k < a - 1$, it follows from (4.6) that

$$m^k_{n-a,n+a} = (2n+1)\int_{-1}^{1} x^{2k} I_{n-a}(x) I_{n+a}(x)\,\mathrm{d}x = 0$$

because all indices in (4.21) and (4.22) have the same parity. We deduce that

$$B_{-a,k} = B_{a,k} = 0 \quad \text{if } k \leqslant a-2\ ,$$

which is consistent with (4.19). We further obtain from (4.7) that

$$B_{0,0} = 1 \quad \text{and} \quad B_{-1,0} = B_{1,0} = -\frac{1}{2}\,,$$

as claimed. Since this fixes the boundary values of our recursion, it suffices to verify that (4.19) satisfies the recurrence relation (4.16). This can be done by observing that the combinatorial numbers $C_{m,l}$ defined, for $m \in \mathbb{N}$ and $l \in \mathbb{N}_0$, by

$$C_{m,l} = \frac{m-2l}{m}\binom{m}{l}$$

satisfy the recurrence relation

$$C_{m+2,l+1} = C_{m,l-1} + 2C_{m,l} + C_{m,l+1} \quad \text{for } m, l \in \mathbb{N}\,, \tag{4.23}$$

see [**20**, Proposition 2.1], and by noting that

$$C_{2k+1,k+a} = \binom{2k}{k+a} - \binom{2k}{k+a-1} \quad \text{and} \quad C_{2k+1,k+a+1} = \binom{2k}{k+a+1} - \binom{2k}{k+a}\,.$$

Thus, the recurrence relation (4.16) is a consequence of (4.23), and we obtain that

$$4^k B_{a,k} = \frac{1}{2}(C_{2k+1,k+a} - C_{2k+1,k+a+1})\,.$$

Finally, we conclude that, for $k \in \mathbb{N}$,

$$B_{0,k} = 4^{-k}\left[\binom{2k}{k} - \frac{1}{2}\left[\binom{2k}{k-1} + \binom{2k}{k+1}\right]\right] = 4^{-k}\left[\binom{2k}{k} - \binom{2k}{k+1}\right] = 4^{-k}C_k\,,$$

which together with $B_{0,0} = 1 = C_0$ establishes (4.20). □

We need one more identity to determine the moments of S_N in the limit $N \to \infty$. This is where the discrepancy between $I_0(x)$ and $\int_{-1}^{x} P_0(z)\,\mathrm{d}z$ becomes useful.

LEMMA 4.7. *For all $k \in \mathbb{N}_0$, we have*

$$\sum_{l=0}^{k}\left(\frac{1}{2l+1} + \frac{1}{2l+3}\right)b_{0,k}^{l} = \frac{2}{2k+1} - \frac{2}{2k+3}\,.$$

Proof. According to the partial fraction decomposition (4.5) in Proposition 4.3, we know

$$m_{0,0}^{k} = -\sum_{l=0}^{k}\left(\frac{1}{2l+1} + \frac{1}{2l+3}\right)b_{0,k}^{l}\,.$$

On the other hand, since $I_0(x) = x - \mathrm{i}$ for $x \in [-1, 1]$, we compute explicitly that

$$m_{0,0}^{k} = \int_{-1}^{1} x^{2k}(I_0(x))^2\,\mathrm{d}x = \int_{-1}^{1} x^{2k}(x-\mathrm{i})^2\,\mathrm{d}x = \frac{2}{2k+3} - \frac{2}{2k+1}\,,$$

and the claimed result follows. □

We can finally describe the moments of S_N in the limit $N \to \infty$.

PROPOSITION 4.8. *For all $k \in \mathbb{N}_0$, we have*

$$\int_{-1}^{1} x^{2k+1} S_N(x)\,\mathrm{d}x = 0 \quad \textit{for all } N \in \mathbb{N}\,,$$

and

$$\lim_{N\to\infty} \int_{-1}^{1} x^{2k} S_N(x)\,\mathrm{d}x = \frac{1}{2}\left(4^{-k} C_k\right)\,.$$

Proof. Using Lemma 4.1 and the odd moments in (4.1) and (4.2), we obtain that

$$\int_{-1}^{1} x^{2k+1} S_N(x)\,\mathrm{d}x = N\left(\frac{2}{2k+3} - \frac{2}{2k+3}\right) = 0 \quad \text{for all } N \in \mathbb{N}\,,$$

as claimed. To determine the limit of the even moments, we fix $k \in \mathbb{N}_0$ and throughout, choose N sufficiently large. For $l \in \mathbb{N}_0$, we rewrite

$$\sum_{n=1}^{N-1}\left(\frac{1}{2n-2l-1} - \frac{1}{2n+2l+3}\right) = \sum_{n=1}^{2l} \frac{1}{2n-2l-1} + \sum_{n=2l+1}^{N-1} \frac{1}{2n-2l-1} - \sum_{n=1}^{N-1} \frac{1}{2n+2l+3}\,,$$

and observe that

$$\sum_{n=1}^{2l} \frac{1}{2n-2l-1} = \sum_{n=1}^{l} \frac{1}{2n-2l-1} + \sum_{n=1}^{l} \frac{1}{2(2l-n+1)-2l-1} = 0$$

as well as

$$\begin{aligned}\sum_{n=2l+1}^{N-1} \frac{1}{2n-2l-1} - \sum_{n=1}^{N-1} \frac{1}{2n+2l+3} &= \sum_{n=1}^{N-2l-1} \frac{1}{2n+2l-1} - \sum_{n=1}^{N-1} \frac{1}{2n+2l+3} \\ &= \frac{1}{2l+1} + \frac{1}{2l+3} - \sum_{n=N-2l-2}^{N-1} \frac{1}{2n+2l+3}\end{aligned}$$

to deduce that

$$\sum_{n=1}^{N-1}\left(\frac{1}{2n-2l-1} - \frac{1}{2n+2l+3}\right) = \frac{1}{2l+1} + \frac{1}{2l+3} - \sum_{n=1}^{2l+2} \frac{1}{2N+2n-2l-3}\,.$$

Applying Proposition 4.3 and rearranging sums further yields

$$\begin{aligned}\sum_{n=1}^{N-1} m_{n,n}^{k} &= \sum_{n=1}^{N-1}\left(\sum_{l=0}^{k} \frac{b_{0,k}^{l}}{2n-2l-1} - \sum_{l=0}^{k} \frac{b_{0,k}^{l}}{2n+2l+3}\right) \\ &= \sum_{l=0}^{k}\sum_{n=1}^{N-1}\left(\frac{1}{2n-2l-1} - \frac{1}{2n+2l+3}\right) b_{0,k}^{l} \\ &= \sum_{l=0}^{k}\left(\frac{1}{2l+1} + \frac{1}{2l+3}\right) b_{0,k}^{l} - \sum_{l=0}^{k}\sum_{n=1}^{2l+2} \frac{b_{0,k}^{l}}{2N+2n-2l-3}\,.\end{aligned}$$

The even moments in (4.1) and (4.2) as well as Lemma 4.7 imply

$$\int_{-1}^{1} x^{2k} S_N(x)\,\mathrm{d}x = N\left(\frac{1}{2k+1} - \frac{1}{2k+3} - \frac{1}{2}\sum_{n=1}^{N-1} m_{n,n}^{k}\right) = \frac{N}{2}\sum_{l=0}^{k}\sum_{n=1}^{2l+2} \frac{b_{0,k}^{l}}{2N+2n-2l-3}\,.$$

Finally, by (4.20) of Lemma 4.6, it follows that

$$\lim_{N\to\infty}\int_{-1}^{1} x^{2k} S_N(x)\,\mathrm{d}x = \frac{1}{2}\sum_{l=0}^{k}\sum_{n=1}^{2l+2}\frac{b_{0,k}^{l}}{2} = \frac{1}{2}\sum_{l=0}^{k}(l+1)b_{0,k}^{l} = \frac{1}{2}B_{0,k} = \frac{1}{2}\left(4^{-k}C_k\right) ,$$

as required. □

The main result of this section is that the moments of S_N converge as $N\to\infty$ to the moments of a scaled semicircle.

Proposition 4.9. *Let $S\colon[-1,1]\to\mathbb{R}$ be given by*

$$S(x) = \frac{1}{\pi}\sqrt{1-x^2} \quad \text{for } x\in[-1,1] .$$

Then, for all $k\in\mathbb{N}_0$, we have

$$\lim_{N\to\infty}\int_{-1}^{1} x^{k} S_N(x)\,\mathrm{d}x = \int_{-1}^{1} x^{k} S(x)\,\mathrm{d}x .$$

Proof. It suffices to show that the moments of S are consistent with Proposition 4.8. Since S is an even function on $[-1,1]$, we certainly have

$$\int_{-1}^{1} x^{2k+1} S(x)\,\mathrm{d}x = 0 \quad \text{for all } k\in\mathbb{N}_0 .$$

Regarding the even moments, we follow [**1**, Section 2.1.1] and make use of the change of variable $x=\sin(\theta)$ where $\theta\in[-\pi/2,\pi/2]$ to write, for $k\in\mathbb{N}_0$,

$$\int_{-1}^{1} x^{2k+2} S(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin^{2k+2}(\theta)\cos^{2}(\theta)\,\mathrm{d}\theta .$$

By integration by parts, we have

$$\int_{-\pi/2}^{\pi/2}\sin^{2k+3}(\theta)\sin(\theta)\,\mathrm{d}\theta = \int_{-\pi/2}^{\pi/2}(2k+3)\sin^{2k+2}(\theta)\cos^{2}(\theta)\,\mathrm{d}\theta ,$$

and using $\cos^2(\theta)=1-\sin^2(\theta)$, we obtain

$$\int_{-1}^{1} x^{2k+2} S(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin^{2k+2}(\theta)\,\mathrm{d}\theta - (2k+3)\int_{-1}^{1} x^{2k+2} S(x)\,\mathrm{d}x .$$

This together with applying integration by parts a second time implies

$$4^{k+1}\int_{-1}^{1} x^{2k+2} S(x)\,\mathrm{d}x = \frac{4^{k+1}}{2k+4}\left(\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin^{2k+2}(\theta)\,\mathrm{d}\theta\right) = \frac{2(2k+1)}{k+2}\left(4^{k}\int_{-1}^{1} x^{2k} S(x)\,\mathrm{d}x\right) .$$

Since

$$\int_{-1}^{1} S(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos^{2}(\theta)\,\mathrm{d}\theta = \frac{1}{2} = \frac{1}{2}C_0$$

and as the Catalan numbers satisfy the recurrence relation

$$C_{k+1} = \frac{2(2k+1)}{k+2}C_k ,$$

it follows that

$$4^{k}\int_{-1}^{1} x^{2k} S(x)\,\mathrm{d}x = \frac{1}{2}C_k ,$$

as needed. □

5. *Fluctuations for iterated Kolmogorov loops*

We establish a Christoffel–Darboux type formula for the integrals of Legendre polynomials and put this together with the asymptotic behaviours discussed in Section 2 as well as the moment analysis performed in Section 4 to prove Theorem 1.5. Using the expression for the iterated Kolmogorov loops given in Proposition 1.3 and determined in Section 3, we finally deduce Theorem 1.6.

The Christoffel–Darboux formula for Legendre polynomials, see [**2**, Remark 5.2.2], which is due to Christoffel [**7**] and Darboux [**8**], states that, for $N \in \mathbb{N}$ and $x, y \in [-1, 1]$,

$$(x-y)\sum_{n=0}^{N}(2n+1)P_n(x)P_n(y) = (N+1)(P_{N+1}(x)P_N(y) - P_N(x)P_{N+1}(y))\ .$$

The second identity in the proposition below can be considered as a Christoffel–Darboux type formula for the integrals of Legendre polynomials.

PROPOSITION 5.1. *Fix $x, y \in [-1, 1]$ and set, for $n \in \mathbb{Z}$,*

$$D_{n+1}(x,y) = I_{n+1}(x)I_n(y) - I_n(x)I_{n+1}(y)\ . \tag{5.1}$$

Then we have

$$(n+2)D_{n+1}(x,y) = (x-y)(2n+1)I_n(x)I_n(y) + (n-1)D_n(x,y)\ , \tag{5.2}$$

and, for all $N \in \mathbb{N}$,

$$(x-y)\sum_{n=1}^{N}(2n+1)I_n(x)I_n(y) = ND_{N+1}(x,y) + 2\sum_{n=1}^{N}D_{n+1}(x,y). \tag{5.3}$$

Proof. Following proof of [**24**, Theorem 3.2.2], we use the recursion formula in Lemma 2.3 to deduce

$$\begin{aligned}
&(n+2)(I_{n+1}(x)I_n(y) - I_n(x)I_{n+1}(y)) \\
&\quad = ((2n+1)xI_n(x) - (n-1)I_{n-1}(x))I_n(y) - I_n(x)((2n+1)yI_n(y) - (n-1)I_{n-1}(y)) \\
&\quad = (x-y)(2n+1)I_n(x)I_n(y) + (n-1)(I_n(x)I_{n-1}(y) - I_{n-1}(x)I_n(y))\ ,
\end{aligned}$$

which establishes (5.2). Applying this identity, we further obtain

$$\begin{aligned}
(x-y)\sum_{n=1}^{N}(2n+1)I_n(x)I_n(y) &= \sum_{n=1}^{N}(n+2)D_{n+1}(x,y) - \sum_{n=1}^{N}(n-1)D_n(x,y) \\
&= ND_{N+1}(x,y) + 2\sum_{n=1}^{N}D_{n+1}(x,y)\ ,
\end{aligned}$$

as claimed. □

This Christoffel–Darboux type formula enters our analysis in the proof of the following lemma.

LEMMA 5.2. *Fix $x, y \in [-1, 1]$. Then, for all $\alpha \in \mathbb{R}$ with $\alpha < 1$, we have*

$$\lim_{N\to\infty}(x-y)N^{\alpha+1}\sum_{n=N}^{\infty}(2n+1)I_n(x)I_n(y) = 0\ .$$

Proof. The result is trivially true if $x \in \{-1, 1\}$ or $y \in \{-1, 1\}$ because $I_n(-1) = I_n(1) = 0$ for all $n \in \mathbb{N}$, cf. (2.13). Let us now suppose that $x, y \in (-1, 1)$ and choose $N, M \in \mathbb{N}$ with

$N < M$. From Proposition 5.1, it follows that

$$(x-y)\sum_{n=N}^{M}(2n+1)I_n(x)I_n(y) = MD_{M+1}(x,y) + 2\sum_{n=N}^{M}D_{n+1}(x,y) - (N-1)D_N(x,y). \quad (5.4)$$

The asymptotic behaviour (2.21) given by the Darboux formula implies that there exists a positive constant $K \in \mathbb{R}$, depending on x and y, such that, for all n sufficiently large,

$$\left|P_n^{(-1,-1)}(x)\right| \leqslant \frac{K}{2}n^{-\frac{1}{2}} \quad \text{and} \quad \left|P_n^{(-1,-1)}(y)\right| \leqslant \frac{K}{2}n^{-\frac{1}{2}} .$$

Since the Jacobi polynomial $P_{n+1}^{(-1,-1)}$ and the integral I_n are related by $P_{n+1}^{(-1,-1)} = \frac{1}{2}nI_n$ for $n \in \mathbb{N}$, see (2.19), we obtain that, for n large enough,

$$|I_n(x)| \leqslant Kn^{-\frac{3}{2}} \quad \text{and} \quad |I_n(y)| \leqslant Kn^{-\frac{3}{2}} .$$

From the definition (5.1) of D_{n+1} we deduce that, for n sufficiently large,

$$|D_{n+1}(x,y)| \leqslant 2K^2n^{-3} . \quad (5.5)$$

In particular, this shows

$$\lim_{M\to\infty} MD_{M+1}(x,y) = 0 ,$$

and, by the integral test, that, for N large enough,

$$\left|\sum_{n=N}^{\infty} D_{n+1}(x,y)\right| \leqslant 2K^2\sum_{n=N}^{\infty}\frac{1}{n^3} \leqslant 2K^2\left(\frac{1}{N^3} + \int_N^{\infty} z^{-3}\,\mathrm{d}z\right) = \frac{2K^2}{N^3} + \frac{K^2}{N^2} .$$

By (5.4), these estimates establish

$$(x-y)\sum_{n=N}^{\infty}(2n+1)I_n(x)I_n(y) = 2\sum_{n=N}^{\infty}D_{n+1}(x,y) - (N-1)D_N(x,y)$$

as well as

$$\left|(x-y)N^{\alpha+1}\sum_{n=N}^{\infty}(2n+1)I_n(x)I_n(y)\right| \leqslant 4K^2N^{\alpha-2} + 2K^2N^{\alpha-1} + N^{\alpha+1}(N-1)|D_N(x,y)| .$$

Provided that $\alpha < 1$, we have $N^{\alpha-1} \to 0$ and $N^{\alpha-2} \to 0$ as $N \to \infty$, and since (5.5) further yields

$$\lim_{N\to\infty} N^{\alpha+1}(N-1)D_N(x,y) = 0 \quad \text{for } \alpha < 1 ,$$

the claimed result follows. □

The reason why the Christoffel–Darboux type formula (5.3) allows us to prove Lemma 5.2 is that as argued in the above proof, the asymptotic (2.21) implies that $D_{n+1}(x,y)$ is of order $O(n^{-3})$ as $n \to \infty$, whereas $(2n+1)I_n(x)I_n(y)$ is only seen to be of order $O(n^{-2})$ as $n \to \infty$.

We use Lemma 5.2 in the proof of Theorem 1.5 to show the convergence away from the diagonal, while the following lemma provides what is needed to establish locally uniform convergence on the diagonal. The convergence of moments, cf. Proposition 4.9, then characterises the limit uniquely.

LEMMA 5.3. *Fix $\varepsilon > 0$. The families*

$$\left\{N \sum_{n=N}^{\infty} (2n+1) I_n(x) I_n(y) \colon N \in \mathbb{N} \text{ and } x, y \in [-1+\varepsilon, 1-\varepsilon]\right\}$$

and

$$\{(N+1) P_N(x) P_{N+1}(x) \colon N \in \mathbb{N} \text{ and } x \in [-1+\varepsilon, 1-\varepsilon]\}$$

are uniformly bounded.

Proof. As a consequence of the estimate (2.21) from the Darboux formula, there exists a positive constant $K \in \mathbb{R}$ such that, for n sufficiently large, we have

$$\left|P_n^{(-1,-1)}(x)\right| \leqslant \frac{K}{2} n^{-\frac{1}{2}} \quad \text{uniformly in } x \in [-1+\varepsilon, 1-\varepsilon] \; .$$

Due to the relation $P_{n+1}^{(-1,-1)} = \frac{1}{2} n I_n$ for $n \in \mathbb{N}$, this implies that, for n large enough,

$$|I_n(x)| \leqslant K n^{-\frac{3}{2}} \quad \text{uniformly in } x \in [-1+\varepsilon, 1-\varepsilon] \; .$$

We deduce that, uniformly in $x, y \in [-1+\varepsilon, 1-\varepsilon]$ and for N sufficiently large,

$$\left| N \sum_{n=N}^{\infty} (2n+1) I_n(x) I_n(y) \right| \leqslant 3NK^2 \sum_{n=N}^{\infty} \frac{1}{n^2} \leqslant 3NK^2 \left(\frac{1}{N^2} + \int_N^{\infty} z^{-2} \, \mathrm{d}z \right) \leqslant 6K^2 \; ,$$

which establishes the uniform boundedness of the first family. We argue in a similar way for the second family. By the asymptotic (2.20) obtained from the Darboux formula and since $P_n^{(0,0)} = P_n$ for $n \in \mathbb{N}_0$, see (2.18), there exists a positive constant $L \in \mathbb{R}$ such that, for N sufficiently large,

$$|P_N(x)| \leqslant L N^{-\frac{1}{2}} \quad \text{uniformly in } x \in [-1+\varepsilon, 1-\varepsilon] \; .$$

Thus, for N large enough,

$$|(N+1) P_N(x) P_{N+1}(x)| \leqslant 2L^2 \quad \text{uniformly in } x \in [-1+\varepsilon, 1-\varepsilon] \; ,$$

and the uniform boundedness of the second family follows. □

We finally combine our results to give the proof of Theorem 1.5.

Proof of Theorem 1.5. As argued for the shifted Legendre polynomials in the proof of Theorem 1.4 in Section 3, the polarised Parseval identity shows that, for $x, y \in [-1, 1]$,

$$\min(1+x, 1+y) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \int_{-1}^{x} P_n(z) \, \mathrm{d}z \int_{-1}^{y} P_n(z) \, \mathrm{d}z \; .$$

Therefore, $R_N(x, y)$ defined by (1.2) can be expressed as, for $N \in \mathbb{N}$,

$$R_N(x, y) = N \sum_{n=N}^{\infty} \frac{2n+1}{2} \int_{-1}^{x} P_n(z) \, \mathrm{d}z \int_{-1}^{y} P_n(z) \, \mathrm{d}z = \frac{1}{2} N \sum_{n=N}^{\infty} (2n+1) I_n(x) I_n(y) \; . \tag{5.6}$$

Hence, if $x, y \in [-1, 1]$ with $x \neq y$, then Lemma 5.2 applied for $\alpha = 0$ implies that $R_N(x, y) \to 0$ as $N \to \infty$, which establishes the desired convergence away from the diagonal. It remains to consider the diagonal case $x = y$. As in Section 4, we consider the functions $S_N \colon [-1, 1] \to \mathbb{R}$ defined by

$$S_N(x) = R_N(x, x) \quad \text{for } x \in [-1, 1] \; .$$

Using the expression for S_N, which follows from (1.2) for R_N, that

$$S_N(x) = N\left(1 + x - \frac{1}{2}(1+x)^2 - \sum_{n=1}^{N-1} \frac{2n+1}{2}\left(\int_{-1}^{x} P_n(z)\,\mathrm{d}z\right)^2\right),$$

and the relation (2.2), we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} S_N(x) &= N\left(1 - (1+x) - \sum_{n=1}^{N-1}(2n+1)P_n(x)\int_{-1}^{x} P_n(z)\,\mathrm{d}z\right)\\ &= -N\left(x + \sum_{n=1}^{N-1} P_n(x)(P_{n+1}(x) - P_{n-1}(x))\right)\\ &= -NP_{N-1}(x)P_N(x)\,.\end{aligned}$$

By Lemma 5.3, it follows that the sequence $(S_N)_{N\in\mathbb{N}}$ is uniformly bounded and uniformly Lipschitz on $[-1+\varepsilon, 1-\varepsilon]$ for $\varepsilon > 0$. The Arzelà–Ascoli theorem implies that $(S_N)_{N\in\mathbb{N}}$ is locally uniformly convergent on $(-1,1)$ and we deduce that $(S_N)_{N\in\mathbb{N}}$ converges to a continuous function on $(-1,1)$. Thus, the limit function is uniquely identified by Proposition 4.9 and since $S_N(-1) = S_N(1) = 0$ for all $N \in \mathbb{N}$, we conclude that, for all $x \in [-1,1]$,

$$\lim_{N\to\infty} R_N(x,x) = \lim_{N\to\infty} S_N(x) = S(x) = \frac{1}{\pi}\sqrt{1-x^2}\,,$$

as required. □

We obtain Theorem 1.6 as a consequence of Theorem 1.5.

Proof of Theorem 1.6. As established in the proof of Lemma 3.2, the covariance function of the process $(L_t^N)_{t\in[0,1]}$ is C_N and hence, the fluctuation process $(F_t^N)_{t\in[0,1]}$ defined by $F_t^N = \sqrt{N}L_t^N$ has covariance NC_N. Moreover, for $s,t \in [0,1]$, we have

$$\begin{aligned}R_N(2s-1, 2t-1) &= 2N\left(\min(s,t) - \sum_{n=0}^{N-1}(2n+1)\int_0^s Q_n(r)\,\mathrm{d}r\int_0^t Q_n(r)\,\mathrm{d}r\right)\\ &= 2NC_N(s,t)\,.\end{aligned} \tag{5.7}$$

By Theorem 1.5, it follows that, for $s,t \in [0,1]$ fixed,

$$\lim_{N\to\infty} NC_N(s,t) = \frac{1}{2}\lim_{N\to\infty} R_N(2s-1,2t-1) = \begin{cases}\frac{1}{\pi}\sqrt{t(1-t)} & \text{if } s = t\\ 0 & \text{if } s \neq t\end{cases}.$$

Thus, for any $k \in \mathbb{N}$ and any $t_1, \ldots, t_k \in [0,1]$, the characteristic functions of the Gaussian random vectors $(F_{t_1}^N, \ldots, F_{t_k}^N)$ converge pointwise as $N \to \infty$ to the characteristic function of the Gaussian random vector $(F_{t_1}, \ldots, F_{t_k})$. By Lévy's continuity theorem, this implies the claimed convergence in finite-dimensional distributions. □

We close with the observation that a slightly modified analysis even allows us to deduce a non-trivial bound on the scale of the decorrelation.

Remark 5.4. Fix $x \in (-1,1)$. For $\beta \in \mathbb{R}$ with $\beta > 0$ and $y \in \mathbb{R}\setminus\{0\}$, set

$$y_N = x + N^{-\beta}y\,.$$

We note that the sequence $(y_N)_{N\in\mathbb{N}}$ converges monotonically to x as $N \to \infty$. Since the asymptotic estimate (2.21) is uniform in $\theta \in [\varepsilon, \pi-\varepsilon]$ for $\varepsilon > 0$, the argument presented to prove Lemma 5.2 can be improved to show the existence of a positive constant $K \in \mathbb{R}$ such that, for n and N large enough,

$$|D_{n+1}(x, y_N)| \leqslant 2K^2 n^{-3} .$$

As in the proof of Lemma 5.2, this implies

$$\left|(x - y_N)N \sum_{n=N}^{\infty} (2n+1) I_n(x) I_n(y_N)\right| \leqslant \frac{4K^2}{N^2} + \frac{4K^2}{N} \leqslant \frac{8K^2}{N} ,$$

and therefore, according to (5.6), that

$$|R_N(x, y_N)| \leqslant \frac{4K^2}{|x - y_N|N} = \frac{4K^2}{|y|} N^{\beta-1} .$$

Hence, as long as $\beta < 1$, we are guaranteed that

$$\lim_{N\to\infty} R_N(x, y_N) = 0 .$$

Due to (5.7), this rewrites in terms of the covariance function C_N, for $s \in (0, 1)$ and $t \in \mathbb{R} \setminus \{0\}$, as

$$\lim_{N\to\infty} NC_N\big(s, s + N^{-\beta}t\big) = \frac{1}{2} \lim_{N\to\infty} R_N(2s - 1, 2s - 1 + N^{-\beta}2t) = 0 \quad \text{for } \beta < 1 ,$$

which provides a bound on the decorrelation scale for the fluctuation processes $(F_t^N)_{t\in[0,1]}$.

Appendix. *Recurrence relations for the partial fraction coefficients*

We continue the analysis started in the proof of Proposition 4.3 to determine recurrence relations for the coefficients $b^l_{a,k}$ and to include them for completeness. Due to the symmetry property (4.3) of Proposition 4.3, we restrict our attention to the family $\{b^l_{a,k} \in \mathbb{R} \colon a, k, l \in \mathbb{N}_0\}$. As discussed when settling the base case for the inductive proof of Proposition 4.3, we have, cf. (4.7),

$$b^0_{0,0} = 1 , \quad b^0_{1,0} = -\frac{1}{2} \quad \text{and} \quad b^l_{a,0} = 0 \text{ otherwise} . \tag{A.1}$$

These are the initial conditions for our recursion. The recurrence relations for the coefficients $b^l_{a,k}$ are deduced, by use of the Heaviside cover-up method, from the expression, for $a \in \mathbb{N}_0$,

$$\begin{aligned} m^k_{n-a,n+a} = {}& \frac{(2n+1)(n-a+2)(n+a+2)}{(2n-2a+1)(2n+2a+1)(2n+3)} \left(\sum_{l=0}^{k-1} \frac{b^l_{a,k-1}}{2n-2l+1} - \sum_{l=0}^{k-1} \frac{b^l_{a,k-1}}{2n+2l+5} \right) \\ & + \frac{(2n+1)(n-a-1)(n+a-1)}{(2n-2a+1)(2n+2a+1)(2n-1)} \left(\sum_{l=0}^{k-1} \frac{b^l_{a,k-1}}{2n-2l-3} - \sum_{l=0}^{k-1} \frac{b^l_{a,k-1}}{2n+2l+1} \right) \\ & + \frac{(n-a+2)(n+a-1)}{(2n-2a+1)(2n+2a+1)} \left(\sum_{l=0}^{k-1} \frac{b^l_{a-1,k-1}}{2n-2l-1} - \sum_{l=0}^{k-1} \frac{b^l_{a-1,k-1}}{2n+2l+3} \right) \\ & + \frac{(n-a-1)(n+a+2)}{(2n-2a+1)(2n+2a+1)} \left(\sum_{l=0}^{k-1} \frac{b^l_{a+1,k-1}}{2n-2l-1} - \sum_{l=0}^{k-1} \frac{b^l_{a+1,k-1}}{2n+2l+3} \right) , \end{aligned} \tag{A.2}$$

which is a consequence of Lemma 4.2 applied with $p = n - a$ and $q = n + a$ and Proposition 4.3. When employing the Heaviside cover-up method, we need to be careful about factors which could occur quadratically in the denominators. By the partial fraction decomposition (4.5) for $m^k_{n-a,n+a}$ we are justified to write

$$b^l_{a,k} = (2n-2l-1)m^k_{n-a,n+a}\big|_{n=l+1/2} \,. \tag{A.3}$$

If $l \neq 0$ and $l \neq a-1$, it follows from (A.2) and (A.3) that, for $k \in \mathbb{N}$,

$$\begin{aligned} b^l_{a,k} = {} & \frac{(l+1)(2l-2a+5)(2l+2a+5)}{16(l+2)(l-a+1)(l+a+1)} b^{l+1}_{a,k-1} + \frac{(l+1)(2l-2a-1)(2l+2a-1)}{16l(l-a+1)(l+a+1)} b^{l-1}_{a,k-1} \\ & + \frac{(2l-2a+5)(2l+2a-1)}{16(l-a+1)(l+a+1)} b^l_{a-1,k-1} + \frac{(2l-2a-1)(2l+2a+5)}{16(l-a+1)(l+a+1)} b^l_{a+1,k-1} \,. \end{aligned}$$

For $l=0$, we need to treat the two cases $a=1$ and $a \neq 1$ separately. If $a \neq 1$, we obtain in the same way as before that

$$\begin{aligned} b^0_{a,k} = {} & \frac{(2a-5)(2a+5)}{32(a-1)(a+1)} b^1_{a,k-1} - \frac{(2a+1)(2a-1)}{8(a-1)(a+1)} \sum_{l=0}^{k-1} \frac{b^l_{a,k-1}}{l+1} \\ & + \frac{(2a-5)(2a-1)}{16(a-1)(a+1)} b^0_{a-1,k-1} + \frac{(2a+1)(2a+5)}{16(a-1)(a+1)} b^0_{a+1,k-1} \,, \end{aligned} \tag{A.4}$$

which for $a=0$ reduces to

$$b^0_{0,k} = \frac{25}{32} b^1_{0,k-1} - \frac{1}{8} \sum_{l=0}^{k-1} \frac{b^l_{0,k-1}}{l+1} - \frac{5}{8} b^0_{1,k-1} \,.$$

If $a=1$, we use the property (4.12) established as part of the proof of Proposition 4.3 to show that the blow-up term $(2n-1)^{-1}$ appearing in the Heaviside cover-up method vanishes, and we deduce

$$\begin{aligned} b^0_{1,k} = {} & \frac{21}{32} \left(\frac{b^0_{1,k-1}}{3} - \frac{b^1_{1,k-1}}{8} - \sum_{l=2}^{k-1} \frac{(l+1) b^l_{1,k-1}}{(l-1)(l+3)} \right) \\ & - \frac{3}{16} \left(\frac{b^0_{0,k-1}}{4} + \sum_{l=1}^{k-1} \frac{(l+1) b^l_{0,k-1}}{l(l+2)} \right) + \frac{21}{16} \left(\frac{b^0_{2,k-1}}{4} + \sum_{l=1}^{k-1} \frac{(l+1) b^l_{2,k-1}}{l(l+2)} \right) . \end{aligned}$$

It remains to consider the case $l = a-1$ for $a \geqslant 2$. As above, we use the property (4.12) to show that the potential blow-up term $(2n-2a+1)^{-1}$ vanishes, and we derive

$$\begin{aligned} b^{a-1}_{a,k} = {} & -\frac{3(4a+3)}{16(a+1)} \left(\frac{b^a_{a,k-1}}{4(a+1)} + \sum_{l=0, l \neq a}^{k-1} \frac{(l+1) b^l_{a,k-1}}{(l-a)(l+a+2)} \right) \\ & + \frac{3(4a-3)}{16(a-1)} \left(\frac{b^{a-2}_{a,k-1}}{4(a-1)} + \sum_{l=0, l \neq a-2}^{k-1} \frac{(l+1) b^l_{a,k-1}}{(l+a)(l-a+2)} \right) \\ & - \frac{3(4a-3)}{16a} \left(\frac{b^{a-1}_{a-1,k-1}}{4a} + \sum_{l=0, l \neq a-1}^{k-1} \frac{(l+1) b^l_{a-1,k-1}}{(l-a+1)(l+a+1)} \right) \\ & + \frac{3(4a+3)}{16a} \left(\frac{b^{a-1}_{a+1,k-1}}{4a} + \sum_{l=0, l \neq a-1}^{k-1} \frac{(l+1) b^l_{a+1,k-1}}{(l-a+1)(l+a+1)} \right) . \end{aligned}$$

While especially the recurrence relations for the cases $l = 0$ and $l = a - 1$ are not particularly nice, we have enough relations to uniquely determine the family $\{b_{a,k}^l \in \mathbb{R} \colon a, k, l \in \mathbb{N}_0\}$ of coefficients from (A.1) by recursion over $k \in \mathbb{N}_0$. It is even possible to use these recurrence relations for the coefficients $b_{a,k}^l$ and (4.12) to prove the recurrence relation for the sums $B_{a,k}$ given in Proposition 4.4 by brute force. However, this approach needs a lot of care and is less elegant. Although it could be of interest to investigate if the above recurrence relations could be significantly simplified. For instance, we note that according to (4.4) of Proposition 4.3, for $a \in \mathbb{N}$,

$$\sum_{l=0}^{k-1} \frac{b_{a,k-1}^l}{l+1} = -\frac{b_{0,k-1}^{a-1}}{2a} ,$$

which implies that (A.4) for $a \geqslant 2$ is equivalent to

$$b_{a,k}^0 = \frac{(2a-5)(2a+5)}{32(a-1)(a+1)} b_{a,k-1}^1 + \frac{(2a+1)(2a-1)}{16a(a-1)(a+1)} b_{0,k-1}^{a-1} + \frac{(2a-5)(2a-1)}{16(a-1)(a+1)} b_{a-1,k-1}^0 + \frac{(2a+1)(2a+5)}{16(a-1)(a+1)} b_{a+1,k-1}^0 .$$

We close by remarking that the coefficients $b_{a,k}^l$ can be easily generated using MATHEMATICA by assigning the appropriate values to `a` and `k`, calling the command

```
Apart[FindSequenceFunction[Table[
  (2n+1)*Integrate[x^(2k)*Integrate[LegendreP[n-a,z],z,-1,x]*
   Integrate[LegendreP[n+a,z],z,-1,x],x,-1,1],n,a+1,a+20],n-a]]
```

and reading off the coefficients $b_{a,k}^l$ for $l \in \{0, \ldots, k\}$, where the upper bound of `n` needs to be increased for large values of `k`.

Acknowledgements. I am grateful to Martin Huesmann and James Norris for helpful discussions.

References

1. G. W. ANDERSON, A. GUIONNET and O. ZEITOUNI, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics 118 (Cambridge University Press, Cambridge, 2010).
2. G. E. ANDREWS, R. ASKEY and R. ROY, *Special functions*, Encyclopedia of Mathematics and its Applications 71 (Cambridge University Press, Cambridge, 1999).
3. G. B. ARFKEN and H. J. WEBER, *Mathematical methods for physicists*, 6th edn (Elsevier, Amsterdam, 2005).
4. F. BAUDOIN, *An introduction to the geometry of stochastic flows* (Imperial College Press, London, 2004).
5. R. BELINSKY, 'Integrals of Legendre polynomials and solution of some partial differential equations', *J. Appl. Anal.* 6 (2000) 259–282.
6. V. I. BOGACHEV, *Gaussian measures*, Mathematical Surveys and Monographs 62 (American Mathematical Society, Providence, RI, 1998).
7. E. B. CHRISTOFFEL, 'Über die Gaussische Quadratur und eine Verallgemeinerung derselben', *J. reine angew. Math.* 55 (1858) 61–82.
8. G. DARBOUX, 'Mémoire sur l'approximation des fonctions de trés-grands nombres, et sur une classe étendue de développements en série', *J. Math. Pures Appl.* (9) 4 (1878) 5–56.
9. J. FOSTER, T. LYONS and H. OBERHAUSER, 'An optimal polynomial approximation of Brownian motion', *SIAM J. Numer. Anal.* 58 (2020) 1393–1421.
10. K. HABERMANN, 'An explicit formula for the inverse of a factorial Hankel matrix', Preprint, arXiv:1808.02880, 2018.
11. K. HABERMANN, 'Geometry of sub-Riemannian diffusion processes', PhD Thesis, University of Cambridge, Cambridge, 2018. Available through the University of Cambridge Repository Apollo. https://doi.org/10.17863/CAM.18862.
12. K. HABERMANN, 'Small-time fluctuations for the bridge in a model class of hypoelliptic diffusions of weak Hörmander type', *Electron. J. Probab.* 24 (2019) 11.
13. G. KALLIANPUR, *Stochastic filtering theory*, Applications of Mathematics 13 (Springer, Berlin, 1980).

14. A. Kolmogoroff, 'Zufällige Bewegungen (zur Theorie der Brownschen Bewegung)', *Ann. of Math.* (2) 35 (1934) 116–117.
15. J. Lamperti, 'On limit theorems for Gaussian processes', *Ann. Math. Statist.* 36 (1965) 304–310.
16. G. F. Lawler and W. Werner, 'The Brownian loop soup', *Probab. Theory Related Fields* 128 (2004) 565–588.
17. N. N. Lebedev, *Special functions and their applications*, Revised English edn (Prentice Hall, Upper Saddle River, NJ, 1965). (Translated and edited by Richard A. Silverman.)
18. M. Loève, *Probability theory II*, 4th edn, Graduate Texts in Mathematics 46 (Springer, Berlin, 1978).
19. J. Mercer, 'XVI, functions of positive and negative type, and their connection with the theory of integral equations', *Philos. Trans. Roy. Soc. London Ser. A* 209 (1909) 415–446.
20. P. J. Miana, H. Ohtsuka and N. Romero, 'Sums of powers of Catalan triangle numbers', *Discrete Math.* 340 (2017) 2388–2397.
21. P. J. Miana and N. Romero, 'Moments of combinatorial and Catalan numbers', *J. Number Theory* 130 (2010) 1876–1887.
22. A. Nica and R. Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series 335 (Cambridge University Press, Cambridge, 2006).
23. L. W. Shapiro, 'A Catalan triangle', *Discrete Math.* 14 (1976) 83–90.
24. G. Szegő, *Orthogonal polynomials*, 4th edn, Colloquium Publications 23 (American Mathematical Society, Providence, RI, 1975).
25. N. Trefethen, 'Brownian paths and random polynomials', version June, 2019, https://www.chebfun.org/examples/stats/RandomPolynomials.html

Karen Habermann
Department of Statistics
University of Warwick
Coventry CV4 7AL
United Kingdom

karen.habermann@warwick.ac.uk